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Toric morphisms and fibrations of toric Calabi-Yau hypersurfaces

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Abstract

Special fibrations of toric varieties have been used by physicists, e.g. the school of Candelas, to construct dual pairs in the study of Het/F-theory duality. Motivated by this, we investigate in this paper the details of toric morphisms between toric varieties. In particular, a complete toric description of fibers - both generic and non-generic -, image, and the flattening stratification of a toric morphism are given. Two examples are provided to illustrate the discussions. We then turn to the study of the restriction of a toric morphism to a toric hypersurface. The details of this can be understood by the various restrictions of a line bundle with a section that defines the hypersurface. These general toric geometry discussions give rise to a computational scheme for the details of a toric morphism and the induced fibration of toric hypersurfaces therein. We apply this scheme to study the family of complex 4-dimensional elliptic Calabi-Yau toric hypersurfaces that appear in a recent work of Braun-Candelas-dlOssa-Grassi. Some directions for future work are listed in the end.

Key words: heterotic-string/F-theory duality, toric morphism, fibration, primitive cone, relative star, toric Calabi-Yau hypersurface, fibred Calabi-Yau manifold, elliptic Calabi-Yau manifold.

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0 Introduction and outline.

Introduction.

Special fibrations of toric varieties have been used by physicists, e.g. the school of Candelas (cf. [A-K-M-S], [B-C-dIO-G], [B-M], [C-F], [C-P-R], [K-S1], [K-S2], [P-S], and [Ra]), to construct dual pairs in the study of Heterotic-string/F-theory duality. This motivates us to investigate the details of toric morphisms between toric varieties and the induced morphisms from the toric hypersurfaces in the domain toric variety of a toric morphism to the target toric variety. The goal of this paper is to describe these details in terms of toric geometry and then to provide a computational scheme for actual toric computations.

The toric description of the generic fiber of a toric morphism has been discussed extensively e.g. in the works of the school of Candelas. Thus, the first theme of this article is to extend their results to a more general setting that takes care of all the fibers - generic and non-generic alike - in terms of toric geometry. The concept of primitive cones and relative stars with respect to a toric morphism are introduced along the way.

Once the details of a toric morphism are described in terms of toric geometry, our second theme is to understand the details of the induced morphism from a toric hypersurface in the domain toric variety to the target toric variety. This is an important step in understanding fibred toric Calabi-Yau hypersurfaces. It turns out that, by turning the problem to a study of the various restrictions of the line bundle with a section that defines the hypersurface, one can extract certain details of this induced morphism in terms of toric geometry as well.

As long as applications to Heterotic-string/F-theory duality is concerned, the discussions in this article will be of no use if no methods of concrete toric computa-

tions for the various details outlined above can be provided. Thus our third theme in this article is to give a computational scheme to match with the toric geometry discussion. This is demonstrated by working out more details of an example: the toric morphism from a toric 5-variety to a toric 3-variety and the induced elliptic fibration of a 4-dimensional toric Calabi-Yau hypersurfaces from a recent collaborative work of Braun, Candelas, de la Ossa, and Grassi ([B-C-dlO-G]). We expect that the computational scheme presented here together with other works from the school of Candelas, notably the classification by Kreuzer and Skarke of higher dimensional reflexive polytopes that admit fibrations (e.g. [K-S1] and [K-S2]), will provide us with some more concrete understanding of fibred Calabi-Yau manifolds beyond current theory. This will await another work in future.

This article is organized as follows. After giving the references of the necessary backgrounds for physicists in Sec. 1, we study in Sec. 2 the various details of a toric morphism. In particular, a complete toric description of fibers - both generic and non-generic -, image, and the flattening stratification of a toric morphism are worked out in Sec. 2.1. Two examples are provided in Sec. 2.2 to illustrate the discussions. We then turn to the study of the restriction of a toric morphism to a toric hypersurface. After providing some necessary facts about line bundles on a toric variety in Sec. 3.1, we relate in Sec. 3.2 the problem of induced morphism to the equivalent problem of the various restrictions of a line bundle and the section in it that defines the hypersurface and study the latter. These general toric geometry discussions give rise to a computational scheme for the details of a toric morphism and the induced morphism of toric hypersurfaces therein. After recalling some necessary facts from the work of Batyrev in Sec. 4.1, we lay down this scheme in Sec. 4.2 and apply it to study the family of 4-dimensional elliptic Calabi-Yau toric hypersurfaces that was discussed in a recent work of Braun-Candelas-dlOssa-Grassi. The Maple codes that are employed for the computation can be found in the preprint version of this article: [math.AG/0010082](#). Some directions for future work are listed in Sec. 5.

Convention 0.1 [real vs. complex manifolds]. A *real* n -dimensional manifold will be called an *n -manifold* while a *complex* n -dimensional manifold an *n -fold*. Also, a real n -dimensional orbifold will be called an *n -orbifold* and a complex n -dimensional variety will be called an *n -variety*. All the varieties in this article are over \mathbb{C} and by a *point in a variety* we mean a *closed point* of that variety only.

Convention 0.2 [morphism vs. fibrations]. In this article, a *morphism* $f : X \rightarrow Y$ means a general map between varieties X and Y as in algebraic geometry; a *fiber* of f means the point-set pre-image $f^{-1}(y)$ of f at a certain (closed) point $y \in Y$. A morphism $f : X \rightarrow Y$ is called a *fibration* if f is surjective and all its fibers have the same dimension $\dim X - \dim Y$.

Outline.

1. Essential mathematical background for physicists.
2. The toric geometry of toric morphisms.

- 2.1 The toric geometry of toric morphisms.
- 2.2 Examples of toric morphisms and their fibers.
- 3. Induced morphism and fibers for hypersurfaces.
 - 3.1 Preparations.
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- 4. Fibration of Calabi-Yau hypersurfaces via toric morphisms.
 - 4.1 General remarks.
 - 4.2 The computational scheme and a detailed study of an example.
- 5. Remarks on further study.

1 Essential mathematical background for physicists.

Some references of the necessary mathematical backgrounds are provided here for the convenience of physicists. The notations of the basic ingredients of toric geometry are also introduced in this section. Essential necessary facts will be recalled along the way of discussions.

- **Toric geometry.** Excellent expositions are given in [C-K], [Fu], and [Gr]. See also [Da], [Do], [Ew], [G-K-Z3], [Od2], and [Zi]. In this article, we fix the following notations:

Notation:

- $N \cong \mathbb{Z}^n$: a lattice;
- $M = \text{Hom}(N, \mathbb{Z})$: the dual lattice of N ;
- $T_N = \text{Hom}(M, \mathbb{C}^*)$: the (complex) torus associated to N ;
- Σ : a fan in $N_{\mathbb{R}}$;
- $\tau \prec \sigma$: the relation between cones τ and σ that τ is in the face of σ ;
- X_{Σ} : the toric variety associated to Σ ;
- $\Sigma(i)$: the i -skeleton of Σ ;
- U_{σ} : the local affine chart of X_{Σ} associated to σ in Σ ;
- $x_{\sigma} \in U_{\sigma}$: the distinguished point associated to σ ;
- O_{σ} : the T_N -orbit of x_{σ} under the T_N -action on X_{Σ} ;
- $V(\sigma)$: the orbit closure of O_{σ} ;
- N_{σ} : the sublattice of N generated as a subgroup by $\sigma \cap N$;
- $M(\sigma) = \sigma^{\perp} \cap M$;

$\text{Span}_{\mathbb{R}}(\sigma)$: the real vector subspace spanned by σ ;
 interior of σ : the topological interior of σ in $\text{Span}_{\mathbb{R}}(\sigma)$;
 λ_v : the one-parameter subgroup of T_N associated to $v \in N$;
 χ^m : the character of T_N associated to $m \in M$.

- **Cox homogeneous coordinate ring.** See [C-K] and [Cox].
- **Toric Calabi-Yau hypersurfaces.** See [Ba2]. Also [Bo], [B-B], [C-K], and [Hü].
- **Fibred Calabi-Yau spaces.** See [Og1], [Og2], and [Og3]. Also [A-M], [H-L-Y], [K-L-M], [K-L-R-Y] and the various works from the school of Candelas, e.g. [A-K-M-S], [B-C-dLO-G], [B-M], [C-F], [C-P-R], [K-S1], [K-S2], [P-S], and [Ra].

2 The toric geometry of toric morphisms.

The toric description of details of a toric morphism is given in Sec. 2.1. Two examples are provided in Sec. 2.2 to illustrate the ideas.

2.1 The toric geometry of toric morphisms.

Recall first the definition:

Definition 2.1.1 [toric morphism]. ([Ew] and [Fu].) Let Σ' , Σ be fans in N' , N respectively. A *map between fans*, in notation $\varphi : \Sigma' \rightarrow \Sigma$, is a homomorphism $\varphi : N' \rightarrow N$ of lattices that satisfies the condition: For each $\sigma' \in \Sigma'$, there exists a $\sigma \in \Sigma$ such that $\varphi(\sigma') \subset \sigma$. Such φ determines a morphism $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma}$. A morphism between toric varieties that arises in this way is called a *toric morphism*. (Cf. FIGURE 2-1-1.)

The following two facts, adapted from [Ew] and [Ei], are fundamental to understanding toric morphisms.

Fact 2.1.2 [equivariance]. (Cf. Theorem 6.4 in [Ew].) *The toric morphism $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma}$ is equivariant with respect to the homomorphism $T_{N'} \rightarrow T_N$ induced by $\varphi : N' \rightarrow N$.*

Fact 2.1.3 [semicontinuity of fiber dimension]. (Cf. Theorem 14.8 in [Ei].) Let $\varphi : X \rightarrow Y$ be a morphism between projective varieties and $s = \max\{\dim X - \dim Y, 0\}$. Then there is a finite filtration of Y by subvarieties (which need not be all distinct),

$$Y \supset \varphi(X) = Y_s \supset Y_{s+1} \supset \cdots \supset Y_{\dim X},$$

such that $\dim \varphi^{-1}(y) \geq i$ if and only if $y \in Y_i$.

The main goal of this subsection is to prove Proposition 2.1.4 below, which extends both a result in [A-K-M-S] related to reflexive polytopes and the usual description of toric bundles as in e.g. [Ew] and [Fu] to full generality. It contains the toric analogue of Fact 2.1.3.

Toric morphism: Image, fibers, and a flattening stratification.

Proposition 2.1.4 [toric morphism]. Let $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma}$ be a toric morphism induced by a map of fans $\varphi : \Sigma' \rightarrow \Sigma$. Then:

- The image $\tilde{\varphi}(X_{\Sigma'})$ of $\tilde{\varphi}$ is a subvariety of X_{Σ} . It is realized as the toric variety associated to the fan $\Sigma_{\varphi} \doteq \Sigma \cap \varphi(N'_{\mathbb{R}})$.
- The fiber of $\tilde{\varphi}$ over a point $y \in X_{\Sigma'}$ depends only on the orbit O_{σ} , $\sigma \in \Sigma_{\varphi}$, that contains y . Denote this fiber by F_{σ} , then it can be described as follows.

Define Σ'_{σ} to be the set of cones σ' in Σ' , whose interior is mapped to the interior of σ . Let $\text{Ind}(\sigma)$ be the index of $\tilde{\varphi}$ over O_{σ} (cf. Definition/Lemma 2.1.7). Then $\varphi^{-1}(y) = F_{\sigma}$ is a disjoint union of $\text{Ind}(\sigma)$ identical copies of connected reducible toric variety F_{σ}^c , whose irreducible components $F_{\sigma}^{\tau'}$ are the toric variety associated to the relative star $\text{Star}_{\sigma}(\tau')$ of the primitive elements τ' in Σ'_{σ} (cf. Definition 2.1.9, Definition 2.1.10, and Lemma 2.1.11).

(FIGURE 2-1-2.)

- For $\sigma \in \Sigma_{\varphi}$, $\tilde{\varphi}^{-1}(O_{\sigma}) = \widetilde{O}_{\sigma} \times F_{\sigma}^c$, where \widetilde{O}_{σ} is a connected covering space of O_{σ} of order $\text{Ind}(\sigma)$. Thus, the stratification of X_{Σ} by $X_{\Sigma} - \tilde{\varphi}(X_{\Sigma'})$ and the toric orbits O_{σ} , $\sigma \in \Sigma_{\varphi}$, of $X_{\Sigma'}$ in $\tilde{\varphi}(X_{\Sigma'})$ gives a flattening stratification of $\tilde{\varphi}$.

Remark 2.1.5. Here the term *reducible toric variety* means a reducible variety obtained by gluing a collection of toric varieties along some isomorphic toric orbits.

Proof of Proposition 2.1.4. Let $N_{\varphi} = N \cap \varphi(N'_{\mathbb{R}})$. Then the toric morphism $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma}$ is the composition of toric morphisms $X_{\Sigma'} \xrightarrow{\tilde{\varphi}} X_{\Sigma_{\varphi}} \xrightarrow{\tilde{l}} X_{\Sigma}$ induced

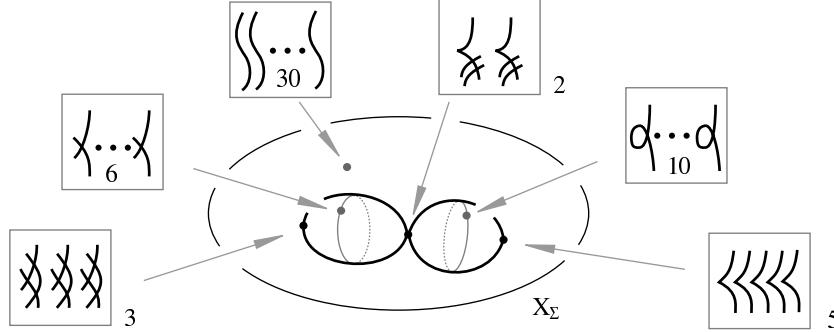


FIGURE 2-1-2. How the various fibers and their index of a toric morphism may look like is indicated. Though they could be wild in general, all the information is coded in the map between fans.

from the maps of fans $\Sigma' \xrightarrow{\varphi} \Sigma_\varphi \xrightarrow{\iota} \Sigma$ from linear maps $N' \xrightarrow{\varphi} N_\varphi \xrightarrow{\iota} N$. Note that $\varphi(N')$ is a subgroup of N_φ of finite index; hence, $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma_\varphi}$ is surjective. On the other hand, since $\Sigma_\varphi = \Sigma \cap N_\varphi \mathbb{R}$, $\tilde{\iota}(X_{\Sigma_\varphi})$ is the closure of the T_{N_φ} -orbit $T_{N_\varphi} \cdot x_{\{0\}}$ in X_Σ . Since it contains T_{N_φ} as an T_{N_φ} -equivariant open dense subset, from the relation between torus actions and fans (e.g. [Fu]) one concludes that the closure is simply X_{Σ_φ} . This shows that $\tilde{\iota} : X_{\Sigma_\varphi} \rightarrow X_\Sigma$ is an inclusion. Together, one concludes that $\tilde{\varphi}(X_{\Sigma'})$ is a subvariety of X_Σ and it is realized as X_{Σ_φ} .

For the remaining part of the proof, we shall regard $\tilde{\varphi}$ as a toric morphism $X_{\Sigma'} \rightarrow X_{\Sigma_\varphi}$ induced by the map of fans $\varphi : \Sigma' \rightarrow \Sigma_\varphi$. After relabelling to simplify the notation, without loss of generality *we shall assume in the following discussion that $\varphi : N'_\mathbb{R} \rightarrow N_\mathbb{R}$ is surjective and, hence $\Sigma_\varphi = \Sigma$.* Note that in this case $\varphi(N')$ is a subgroup of finite index in N and the induced group homomorphism $T_{N'} \rightarrow T_N$ is surjective.

Since $\tilde{\varphi}$ is toric equivariant, it sends a toric orbit $O_{\sigma'}$ in $X_{\Sigma'}$ into a toric orbit O_σ in X_Σ . The following lemma characterizes such pairs (σ', σ) .

Lemma 2.1.6 [orbit pair]. $\tilde{\varphi}$ maps $O_{\sigma'}$ to O_σ if and only if φ maps the interior of σ' to the interior of σ . When this happens, $\tilde{\varphi}$ maps $O_{\sigma'}$ surjectively to O_σ .

Proof. The if-part follows from the natural presentation of orbits as affine subvarieties in a toric variety ([Fu]): $O_{\sigma'} = \text{Spec}(\mathbb{C}[\sigma'^\perp \cap M'])$ and $O_\sigma = \text{Spec}(\mathbb{C}[\sigma^\perp \cap M])$. Note that, for each $\sigma' \in \Sigma'$, there is a unique $\sigma \in \Sigma$ such that the interior of σ' is mapped to the interior of σ . Thus, $\tilde{\varphi}$ maps $O_{\sigma'}$ to a particular O_σ determined by φ . Since X_Σ is stratified by toric orbits and the image of an orbit in $X_{\Sigma'}$ can

be contained only in a unique orbit in X_Σ , the only-if-part also follows. For such (σ', σ) , the surjectivity of $\tilde{\varphi} : O_{\sigma'} \rightarrow O_\sigma$ follows from the surjectivity of $T_{N'} \rightarrow T_N$. This concludes the proof.

□

To understand the fiber over each orbit, consider a homomorphism between lattices $\psi : N_1 \rightarrow N_2$ with $\psi(N_1)$ of finite index in N_2 . Let $[N_2 : \psi(N_1)]$ be the index of $\psi(N_1)$ in N_2 and let $\tilde{\psi} : T_{N_1} \rightarrow T_{N_2}$ be the induced morphism, where T_{N_1} , T_{N_2} are regarded as the toric variety associated to the fans that consist only of the origin. One has then the following exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \psi^{-1}(0) & \longrightarrow & N_1 & \longrightarrow & \psi(N_1) \\ & & 0 & \longrightarrow & \psi(N_1) & \longrightarrow & N_2 & \longrightarrow & N_2/\psi(N_1) \\ & & & & & & & & \longrightarrow 0 \end{array} ,$$

Note that $\psi^{-1}(0)$ is a lattice of rank $\text{rank } N_1 - \text{rank } N_2$ and $N_2/\psi(N_1)$ is a finite abelian group. From the first exact sequence, the homomorphism $T_{N_1} \rightarrow T_{\psi(N_1)}$ is a bundle map with fiber $T_{\psi^{-1}(0)}$. From the second exact sequence, the homomorphism $T_{\psi(N_1)} \rightarrow T_{N_2}$ is a finite covering map with fiber a finite set of $[N_2 : \psi(N_1)]$ -many elements. Since the composition of these two morphisms gives the morphism $\tilde{\psi} : T_{N_1} \rightarrow T_{N_2}$, one concludes that the fiber of $\tilde{\psi}$ is a disjoint union of $[N_2 : \psi(N_1)]$ copies of $T_{\psi^{-1}(0)}$. Notice that the same conclusion can also be phrased in terms of lattices M_1, M_2 dual to N_1, N_2 respectively.

Apply this to our problem. Let Σ'_σ be the set of cones in Σ' whose interior is mapped to the interior of $\sigma \in \Sigma$. Let $\sigma' \in \Sigma'_\sigma$. Since $\tilde{\varphi} : O_{\sigma'} \rightarrow O_\sigma$ is surjective, the image of the homomorphism behind, $\overline{\varphi}_{\sigma'}^{\sigma'} : N'/N'_{\sigma'} \rightarrow N/N_\sigma$, must be of finite index in N/N_σ . (Equivalently, the induced homomorphism $\varphi_{\sigma'}^\dagger : \sigma^\perp \cap M \rightarrow (\sigma')^\perp \cap M'$ must be injective.) Since

$$\overline{\varphi}_{\sigma'}^{\sigma'-1}(0) = \varphi^{-1}(N_\sigma)/N'_{\sigma'},$$

the fiber $F_\sigma^{\sigma'}$ of $\tilde{\varphi}$ over a point in O_σ is thus a disjoint union of $[N/N_\sigma : \overline{\varphi}_{\sigma'}^{\sigma'}(N'/N'_{\sigma'})]$ copies of $T_{\varphi^{-1}(N_\sigma)/N'_{\sigma'}}$.

The following two lemmas play an essential role in understanding fibers of $\tilde{\varphi}$ and their relations.

Definition/Lemma 2.1.7 [index of $\tilde{\varphi}$ over O_σ]. *The image $\overline{\varphi}_{\sigma'}^{\sigma'}(N'/N'_{\sigma'})$ in N/N_σ is independent of the choice of σ' in Σ'_σ . Thus, the index $[N/N_\sigma : \overline{\varphi}_{\sigma'}^{\sigma'}(N'/N'_{\sigma'})]$ depends only on σ and will be denoted by $\text{Ind}(\sigma)$ and called the index of $\tilde{\varphi}$ over O_σ .*

Proof. Let $\sigma'_1, \sigma'_2 \in \Sigma'_\sigma$. Assume first that σ'_1 is in the face of σ'_2 . Then $N'_{\sigma'_1}$ is a subspace of $N'_{\sigma'_2}$ and one has the exact sequence :

$$0 \longrightarrow N'_{\sigma'_2}/N'_{\sigma'_1} \longrightarrow N'/N'_{\sigma'_1} \longrightarrow N'/N'_{\sigma'_2} \longrightarrow 0.$$

Note that since $N'_{\sigma'_2}$ is mapped by $\overline{\varphi}_{\sigma}^{\sigma'_2}$ to the 0 in N/N_{σ} , $N'_{\sigma'_2}/N'_{\sigma'_1}$ as a subspace in $N'/N'_{\sigma'_1}$ is mapped by $\overline{\varphi}_{\sigma}^{\sigma'_1}$ also to the 0 in N/N_{σ} . Since $N'/N'_{\sigma'_2} = (N'/N'_{\sigma'_1})/(N'_{\sigma'_2}/N'_{\sigma'_1})$ canonically, $\overline{\varphi}_{\sigma}^{\sigma'_1}$ descends to $\overline{\varphi}_{\sigma}^{\sigma'_2}$ also canonically after taking the quotient by $N'_{\sigma'_2}/N'_{\sigma'_1}$. This shows that $\overline{\varphi}_{\sigma}^{\sigma'_1}(N'/N'_{\sigma'_1}) = \overline{\varphi}_{\sigma}^{\sigma'_2}(N'/N'_{\sigma'_2})$ if σ'_1 is in the face of σ'_2 .

For general $\sigma'_1, \sigma'_2 \in \Sigma'_{\sigma}$, let $\overset{\circ}{\sigma}$ be the interior of σ and regard φ as a map $N'_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$. Then, since $\varphi^{-1}(\overset{\circ}{\sigma})$ is connected, one can find a finite sequence of cones $\sigma'_1, \tau'_1, \dots, \tau'_s, \sigma'_2$ in Σ'_{σ} such that each element in the sequence either contains as a face or is in the face of the next element. By transitivity, this shows that $\overline{\varphi}_{\sigma}^{\sigma'_1}(N'/N'_{\sigma'_1}) = \overline{\varphi}_{\sigma}^{\sigma'_2}(N'/N'_{\sigma'_2})$ and we conclude the proof.

□

Lemma 2.1.8. *If τ is in the face of σ , then $Ind(\tau)$ is an integral multiple of $Ind(\sigma)$.*

Proof. Since τ is in the face of σ , $\varphi^{-1}(\tau)$ must lie in the boundary of $\varphi^{-1}(\sigma)$. Since both $\varphi^{-1}(\tau)$ and $\varphi^{-1}(\sigma)$ are unions of cones, each cone in $\varphi^{-1}(\tau)$, in particular each $\tau' \in \Sigma'_{\tau}$, must lie in the face of some cone σ' in Σ'_{σ} . Consider now the following diagram of abelian groups with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & N'_{\sigma'}/N'_{\tau'} & \longrightarrow & N'/N'_{\tau'} & \longrightarrow & N'/N'_{\sigma'} \\ & & \alpha \downarrow & & \overline{\varphi}_{\tau'}^{\tau'} \downarrow & & \overline{\varphi}_{\sigma}^{\sigma'} \downarrow \\ 0 & \rightarrow & N_{\sigma}/N_{\tau} & \longrightarrow & N/N_{\tau} & \longrightarrow & N/N_{\sigma} & \longrightarrow & 0 . \end{array}$$

Note that the diagram commutes since all the homomorphisms involved are induced from the morphism $\varphi : N' \rightarrow N$. By the Snake Lemma in homological algebra ([Ei]), one has the exact sequence

$$0 \rightarrow \ker \alpha \rightarrow \ker \overline{\varphi}_{\sigma}^{\sigma'} \rightarrow \ker \overline{\varphi}_{\sigma}^{\sigma'} \rightarrow \text{coker } \alpha \rightarrow \text{coker } \overline{\varphi}_{\tau'}^{\tau'} \rightarrow \text{coker } \overline{\varphi}_{\sigma}^{\sigma'} \rightarrow 0 ,$$

which implies that $Ind(\sigma)$ divides $Ind(\tau)$ by considering the cardinality of the last three terms. This concludes the proof.

□

We are now ready to understand how the toroidal pieces from the orbit-to-orbit fibrations are joined together to form a complete fiber.

Let $\sigma \in \Sigma$. By assumption, Σ'_{σ} is non-empty. We shall regard Σ'_{σ} as a subfan in Σ' . Recall the construction of the star of a cone in toric geometry. Here we need a relative version defined as follows.

Let $\tau' \in \Sigma'_\sigma$ and $\{\sigma'_1, \sigma'_2, \dots\}$ be the set of cones in Σ'_σ that contains τ' as a face. Then each σ'_i determines a cone $\overline{\sigma'_i}$ in $\varphi^{-1}((N_\sigma)_\mathbb{R})/(N'_{\tau'})_\mathbb{R}$, defined by

$$\overline{\sigma'_i} = (\sigma'_i + (N'_{\tau'})_\mathbb{R})/(N'_{\tau'})_\mathbb{R}.$$

Note that $\sigma'_i + (N'_{\tau'})_\mathbb{R}$ is contained in $\varphi^{-1}((N_\sigma)_\mathbb{R})$ since $\tau', \sigma'_i \in \Sigma'_\sigma$. Thus, $\{\overline{\sigma'_1}, \overline{\sigma'_2}, \dots\}$ defines a fan in $\varphi^{-1}((N_\sigma)_\mathbb{R})/(N'_{\tau'})_\mathbb{R}$.

Definition 2.1.9 [relative star]. The fan in $\varphi^{-1}((N_\sigma)_\mathbb{R})/(N'_{\tau'})_\mathbb{R}$ constructed above will be called the *relative star* of τ' over σ and will be denoted by $\text{Star}_\sigma(\tau')$.

Definition 2.1.10 [primitive cone]. A cone $\tau' \in \Sigma'_\sigma$ is called *primitive* with respect to φ if none of the faces of τ' are in Σ'_σ . The set of primitive cones in Σ'_σ will be denoted by Σ'°_σ .

(Cf. FIGURE 2-1-3.)

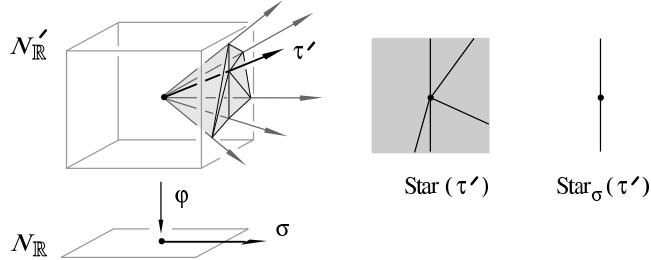


FIGURE 2-1-3. A comparison of the relative star $\text{Star}_\sigma(\tau')$ with respect to a map of fans φ and the usual star $\text{Star}(\tau')$.

The following lemma justifies the two definitions above.

Lemma 2.1.11 [connected/irreducible components of fiber]. Given $y \in O_\sigma$, let F_σ^c be a connected component of $\tilde{\varphi}^{-1}(y)$. Then

- (a) There is a 1 – 1 correspondence between the irreducible components of the fiber F_σ^c over O_σ and the elements in Σ'°_σ .
- (b) Let $\tau' \in \Sigma'^\circ_\sigma$, then the component $F_\sigma^{\tau'}$ of F_σ^c associated to τ' is the toric variety associated to the fan $\text{Star}_\sigma(\tau')$ in $\varphi^{-1}((N_\sigma)_\mathbb{R})/(N'_{\tau'})_\mathbb{R}$. It has (complex) dimension: $\text{codim } \tau' - \text{codim } \sigma$.
- (c) If $\tau'_1, \dots, \tau'_s \in \Sigma'^\circ_\sigma$ generate a cone σ' in Σ' , then $\sigma' \in \Sigma'_\sigma$ and $\bigcap_{i=1}^s F_\sigma^{\tau'_i} = \text{Star}_\sigma(\sigma')$.
- (d) The whole fiber F_σ over a point in O_σ is the disjoint union of $\text{Ind}(\sigma)$ -many copies of F_σ^c .

Proof. (a) *The 1 – 1 correspondence:* By definition, any non-primitive element σ' in Σ'_σ must contain some primitive element τ' in its face. This implies that $F_{\sigma'}^{\sigma'}$ is in the closure of $F_\sigma^{\tau'}$. On the other hand, for a primitive τ' in Σ'_σ the fiber $F_\sigma^{\tau'}$ cannot be contained in the closure of $F_{\sigma'}^{\sigma'}$ for any $\sigma' \neq \tau'$. This shows that $F_\sigma^{\tau'}$ is a maximal stratum in F_σ^c and, hence, its closure must be a component of F_σ^c . Together, these prove the 1 – 1 correspondence.

(b) *The toric geometry of a component:* Let τ' be a primitive element in Σ'_σ . Then the closure $\overline{F_\sigma^{\tau'}}$ consists of $F_\sigma^{\tau'}$ and all $F_{\sigma'}^{\sigma'}$ with σ' containing τ' in its face. By construction, this is exactly the fan $\text{Star}_\sigma(\tau')$ in $\varphi^{-1}((N_\sigma)_\mathbb{R})/(N'_{\tau'})_\mathbb{R}$, which has dimension $\text{codim } \tau' - \text{codim } \sigma$. This proves Part (b).

(c) *The intersection of irreducible components:* This follows from the definition of relative stars and the order-reversing correspondence between cones σ' in Σ'_σ and toric varieties associated to $\text{Star}_\sigma(\sigma')$.

(d) *The number of connected components:* Note that the fiber F_σ over a point in O_σ is naturally stratified by complex tori of the form $(\mathbb{C}^*)^r$. Lemma 2.1.6 implies that each σ' contributes to F_σ exactly $\text{Ind}(\sigma)$ -many strata $(\mathbb{C}^*)^r$, where $r = \text{codim } \sigma' - \text{codim } \sigma_\sigma$. Each connected components of F_σ will contain at least one of them. On the other hand, the restriction of the map $\tilde{\varphi} : \tilde{\varphi}^{-1}(O_\sigma) \rightarrow O_\sigma$ factors through a covering morphism $(\mathbb{C}^*) \rightarrow O_\sigma$ of degree $\text{Ind}(\sigma)$. Together this shows that F_σ must have exactly $\text{Ind}(\sigma)$ -many connected components.

This concludes the proof. □

This justifies the description of fibers of a toric morphism in Proposition 2.1.4.

Finally, let us consider the induced stratification of $\tilde{\varphi}$ by toric orbits O_σ of X_Σ . Observe first the following commutative diagram of maps of fans

$$\begin{array}{ccccc} \Sigma'_{\{0\}} & \hookrightarrow & \Sigma' & = & \Sigma' \\ \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\ \{0\}_{\varphi(N')} & \hookrightarrow & \Sigma_{\varphi(N')} & \rightarrow & \Sigma \end{array},$$

where $\{0\}_{\varphi(N')}$ is the fan in $\varphi(N')_\mathbb{R}$ that consists only of $\{0\}$, $\Sigma'_{\{0\}}$ is regarded as a fan in $N'_\mathbb{R}$, and $\Sigma_{\varphi(N')}$ is the naturally induced fan in $\varphi(N')_\mathbb{R}$ from Σ in $N_\mathbb{R}$. By taking the toric variety associated to the fans in the diagram, one obtains the torically equivariant commutative diagram

$$\begin{array}{ccc} \widetilde{O_{\{0\}}} \times F_{\{0\}}^c & \hookrightarrow & X_{\Sigma'} \\ \downarrow \tilde{\varphi} & & \downarrow \tilde{\varphi} \\ O_{\{0\}} & \hookrightarrow & X_\Sigma \end{array},$$

where $O_{\{0\}}$ is the open dense T_N -orbit in X_Σ and $\widetilde{O}_{\{0\}} \xrightarrow{\tilde{\varphi}} O_{\{0\}}$ is a covering map of order $[N : \varphi(N')] = \text{Ind}(\{0\})$. This shows that $\tilde{\varphi}^{-1}(O_{\{0\}}) = \widetilde{O}_{\{0\}} \times F_{\{0\}}^c$, as claimed. For the stratum over general orbit O_σ , $\sigma \in \Sigma$, let τ' be a primitive element in Σ'_σ . Then the discussion above applied to the following commutative diagram of maps of fans

$$\begin{array}{ccccccc} \text{Star}_\sigma(\tau') & \hookrightarrow & \text{Star}(\tau') & = & \text{Star}(\tau') \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \text{Star}(\sigma) & \rightarrow & \text{Star}(\sigma) \end{array}$$

from the maps of lattices induced by φ (with the fans in the above diagram lying in the realization of the lattices in the diagram below at the same relative positions)

$$\begin{array}{ccccc} N'/N'_{\tau'} & = & N'/N'_{\tau'} & = & N'/N'_{\tau'} \\ \downarrow \overline{\varphi}'_\sigma & & \downarrow \overline{\varphi}'_\sigma & & \downarrow \overline{\varphi}'_\sigma \\ \overline{\varphi}'_\sigma(N'/N'_{\tau'}) & = & \overline{\varphi}'_\sigma(N'/N'_{\tau'}) & \hookrightarrow & N/N_\sigma \end{array}$$

shows that $\widetilde{O}_\sigma \times F_\sigma^{\tau'}$ is contained in $\tilde{\varphi}^{-1}(O_\sigma)$, where $\widetilde{O}_\sigma \rightarrow O_\sigma$ is a covering map of order $\text{Ind}(\sigma)$. From the pasting of the irreducible components $F_\sigma^{\tau'}$ to form F_σ^c discussed earlier, one concludes that $\tilde{\varphi}^{-1}(O_\sigma) = \widetilde{O}_\sigma \times F_\sigma^c$.

This concludes the proof of Proposition 2.1.4.

□

Remark 2.1.12. If $[N : \varphi(N')] < \infty$, then

$$\text{Ind}(\{0\}) = [N : \varphi(N')] = \text{Ind}(\sigma) \cdot [N_\sigma : N_\sigma \cap \varphi(N')].$$

In particular, if φ is surjective, then $\text{Ind}(\sigma) = 1$ for all $\sigma \in \Sigma$.

Characterization of toric fibrations.

To characterize a fibration, let us assume that $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_\Sigma$ is surjective. Let $F_\sigma^{\tau'}$ be an irreducible component of the connected fiber F_σ^c over $O_\sigma \subset X_\Sigma$. Then, from Lemma 2.1.11,

$$\begin{aligned} \dim F_\sigma^{\tau'} &= \dim \left(\varphi^{-1}((N_\sigma)_\mathbb{R}) / (N'_{\tau'})_\mathbb{R} \right) = \dim \varphi^{-1}(0) + \dim (N_\sigma)_\mathbb{R} - \dim (N'_{\tau'})_\mathbb{R} \\ &\geq \dim \varphi^{-1}(0) = \dim F_0 = \dim X_{\Sigma'} - \dim X_\Sigma, \end{aligned}$$

where F_0^c is a connected component of a generic fiber and we use the fact that, for τ' primitive, the map $\varphi_\mathbb{R} : (N'_{\tau'})_\mathbb{R} \rightarrow (N_\sigma)_\mathbb{R}$ is injective. From this, one concludes:

Corollary 2.1.13 [fibration]. *Let $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_\Sigma$ is a surjective toric morphism. Then $\tilde{\varphi}$ is a fibration, in the sense that all the irreducible components of fibers of*

$\tilde{\varphi}$ have the same dimension, $\dim X_{\Sigma'} = \dim X_{\Sigma}$, if and only if the map of cones $\varphi_{\mathbb{R}} : \tau' \rightarrow \sigma$ is a bijection for every $\tau' \in \Sigma'_\sigma^\circ$.

In particular, since a ray in $\Sigma'(1) - \varphi^{-1}(0)$ must be a primitive cone, if $\tilde{\varphi}$ is a fibration, then φ maps $\Sigma'(1)$ to $\Sigma(1)$ surjectively.

Dual picture via polytopes in M lattice.

As the above proposition indicates, the study of toric morphisms via fans in the N lattice is very natural. Nevertheless, for projective toric varieties, it is convenient to have the dual polyhedral picture in the M lattice in mind. We summarize some essential features here. They follow directly from the proof of Proposition 2.1.4 by taking dual and tracing definitions.

Let $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma}$ be a toric morphism between projective toric varieties. Without loss of generality, we may assume that $\varphi_{\mathbb{R}} : N'_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$ is surjective. Note that this implies that $\varphi^{\dagger} : M \rightarrow M'$ is injective. Let Δ' (resp. Δ) be a convex polytope in $M'_{\mathbb{R}}$ whose normal fan is Σ' (resp. Σ). The face of Δ whose normal cone is σ will be denoted by Θ_{σ} . Note that, by definition, the face associated to $\{0\}$ in $N_{\mathbb{R}}$ is Δ . Similarly for the face $\Theta_{\sigma'}$ of Δ' .

The following equivalences play a key role in translating the fan picture to the dual polytope picture:

$$\begin{aligned} & \varphi(\text{interior of } \sigma') \subset \text{interior of } \sigma. \\ \iff & \langle u, \varphi(v') \rangle > 0 \quad \text{for all } u \in \sigma^{\vee} - \sigma^{\perp} \text{ and } v' \text{ in the interior of } \sigma'. \\ \iff & \langle \varphi^{\dagger}(u), v' \rangle > 0 \quad \text{for all } u \in \sigma^{\vee} - \sigma^{\perp} \text{ and } v' \text{ in the interior of } \sigma'. \\ \iff & \varphi^{\dagger}(\sigma^{\vee} - \sigma^{\perp}) \subset \sigma'^{\vee} - \sigma'^{\perp}. \quad (\text{Note that } \varphi^{\dagger}(\sigma^{\perp}) \subset \sigma'^{\perp}.) \end{aligned}$$

For $\sigma \in \Sigma$, define the σ -lighted part $\Delta'_{\sigma} \subset \Delta'$ associated to a face $\Theta_{\sigma} \subset \Delta$ by

$$\Delta'_{\sigma} = \{ \Theta_{\sigma'} \in \Delta' \mid \varphi^{\dagger}(\sigma^{\vee} - \sigma^{\perp}) \subset \sigma'^{\vee} - \sigma'^{\perp} \}.$$

(The terminology is inspired by FIGURE 2-1-4.) Note that, for an interior point $p \in \Theta_{\sigma'}$, the *unridged tangent cone* $T_p \Delta' - T_p \Theta_{\sigma'}$ at p is isomorphic to $\sigma'^{\vee} - \sigma'^{\perp}$. Thus, Δ'_{σ} is the polyhedral sub-complex of Δ' that consists of faces, the unridged tangent cone at an interior point of which contains $\varphi^{\dagger}(\sigma^{\vee} - \sigma^{\perp})$. By definition, $\Theta_{\sigma'} \in \Delta'_{\sigma}$ if and only if the interior of σ' is mapped by φ to the interior of σ . If one regards Δ' as a collection of combinatorial polytopes related by inclusion relations, then Δ' is decomposed into a disjoint union $\sqcup_{\sigma \in \Sigma} \Delta'_{\sigma}$. (FIGURE 2-1-4.)

Lemma 2.1.14 [polytope for relative star]. *Let $\Theta_{\tau'} \in \Delta'_{\sigma}$, regarded as a polytope in τ'^{\perp} . Let $\overline{\varphi^{\dagger}}_{\sigma}^{\tau'} : \tau'^{\perp} \rightarrow \tau'^{\perp}/\varphi^{\dagger}(\sigma^{\perp})$ be the quotient map. Then the normal fan of the polytope $\overline{\Theta}_{\tau'} = \overline{\varphi^{\dagger}}_{\sigma}^{\tau'}(\Theta_{\tau'})$ in $\tau'^{\perp}/\varphi^{\dagger}(\sigma^{\perp})$ is the relative star $\text{Star}_{\sigma}(\tau')$. (FIGURE 2-1-5.)*

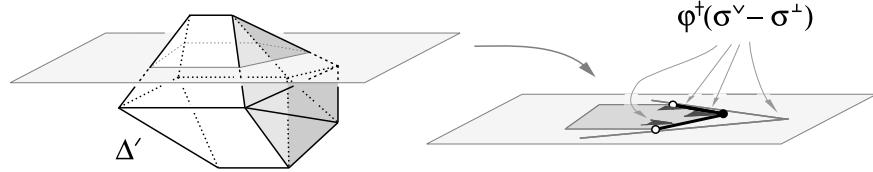


FIGURE 2-1-4. The σ -lighted part $\Delta'_\sigma \subset \Delta'$ associated to a face $\Theta_\sigma \subset \Delta$ is indicated by the shaded polyhedral subcomplex. A cross-section to indicate why these faces are σ -lighted is also indicated. (Cf. Forward light cones, domain of influence, ..., in general relativity.)

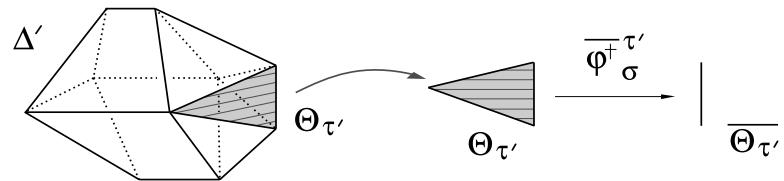


FIGURE 2-1-5. The polytope $\overline{\Theta_{\tau'}}$ in $\tau'^\perp / \varphi^\dagger(\sigma^\perp)$ whose normal fan is the relative star $Star_\sigma(\tau')$ is indicated. (In the picture, $\varphi^\dagger(\sigma^\perp)$ is realized as a foliation of τ'^\perp .)

Proof. Recall Definition 2.1.9 of relative stars. Let $\tau' \in \Sigma'_\sigma$; then there is a non-degenerate pairing $\varphi^{-1}(N_\sigma)/N'_{\tau'} \times \tau'^\perp/\varphi^\dagger(\sigma^\perp) \rightarrow \mathbb{Z}$ induced by the restriction of the canonical pairing $N' \times M' \rightarrow \mathbb{Z}$ to $\varphi^{-1}(N_\sigma) \times \tau'^\perp \rightarrow \mathbb{Z}$. Consequently, if $\sigma' \in \Sigma'_\sigma$ contains τ' in its face, then the dual cone of $\overline{\Theta}' = (\sigma' + (N'_{\tau'})_{\mathbb{R}})/(N'_{\tau'})_{\mathbb{R}} \subset \varphi^{-1}((N_\sigma)_{\mathbb{R}})/(N'_{\tau'})_{\mathbb{R}}$ is simply $((\sigma'^\vee \cap \tau'^\perp) + \varphi^\dagger(\sigma^\perp))/\varphi^\dagger(\sigma^\perp) \subset \tau'^\perp/\varphi^\dagger(\sigma^\perp)$, which is the tangent cone of $\overline{\Theta}_{\tau'}$ at an interior point of the face $\overline{\varphi^\dagger(\sigma)}(\Theta_{\sigma'}) \subset \overline{\Theta}_{\tau'}$. This gives a correspondence from $Star_\sigma(\tau')$ to the set of faces of $\overline{\Theta}_{\tau'}$. This also implies that the normal fan of $\overline{\Theta}_{\tau'}$ contains $Star_\sigma(\tau')$. Since both fans are complete, they must be identical. This concludes the proof. \square

Similarly, one has the dual characterization for primitive elements σ' in Σ'_σ :

Definition 2.1.15 [primitive face]. A face $\Theta_{\sigma'}$ in Δ'_σ is called *primitive* if it is not in the boundary of any other face in Δ'_σ .

One can check that σ' is a primitive element in Σ'_σ if and only if $\Theta_{\sigma'}$ is a primitive face of Δ'_σ . The following corollary follows immediately from Lemma 2.1.11 and Lemma 2.1.14 (cf. the notations in Lemma 2.1.11).

Corollary 2.1.16 [polytope for irreducible component of fiber]. (a) There is a 1 – 1 correspondence between the primitive faces in Δ'_σ and the irreducible components of the fiber F_σ^c of $\tilde{\varphi}$ over a point in O_σ . (b) Let $\Theta_{\tau'}$ be a primitive face of Δ'_σ . Then the corresponding irreducible component $F_\sigma^{\tau'}$ in F_σ^c is the toric variety associated to the normal fan of the polytope $\overline{\Theta}_{\tau'}$ in $\tau'^\perp/\varphi^\dagger(\sigma^\perp)$.

In particular, for the generic fiber of the morphism $\tilde{\varphi}$, $\{0\}^\vee = M$ and $\Delta'_{\{0\}}$ consists of Δ' and all the faces $\Theta_{\sigma'}$ of Δ' with $\sigma'^\perp \supset \varphi^\dagger(M)$. The normal fan of the projection of $\Delta'_{\{0\}}$ in $(M'/\varphi^\dagger(M))_{\mathbb{R}}$ is exactly $\Sigma'_{\{0\}}$ as discussed earlier. (Cf. [A-K-M-S].)

2.2 Examples of toric morphisms and fibers.

In this subsection, we give two examples of toric morphisms and their fibers to illustrate the previous discussions. The second example is a toric fibration from [B-C-dLO-G] and will be the major example for later discussions of this article.

Example 2.2.1 [$\tilde{\varphi} : X_{\Sigma'} \rightarrow \mathbb{CP}^1$]. Let $\Sigma = \{\sigma_+, \{0\}, \sigma_-\}$ be the fan for \mathbb{CP}^1 . Then one has the corresponding decomposition $\Sigma' = \Sigma'_{\sigma_+} \cup \Sigma'_{\{0\}} \cup \Sigma'_{\sigma_-}$. Let $\Sigma'(1) = \Sigma'_+(1) \cup \Sigma'_0(1) \cup \Sigma'_-(1)$ be the corresponding decomposition. Since σ_+ and σ_- are

1-dimensional and cones in Σ' are strongly convex, the set of primitive elements in Σ'_{σ_+} (resp. Σ'_{σ_-}) is exactly $\Sigma'_+(1)$ (resp. $\Sigma'_-(1)$). Also, since σ_{\pm} is a maximal cone, for $\sigma' \in \Sigma'_{\pm}(1)$ the relative star $Star_{\sigma_{\pm}}(\sigma')$ coincides with the usual star $Star(\sigma')$. Consequently, all the irreducible components of fibers of $\tilde{\varphi}$ are of codimension 1 in $X_{\Sigma'}$. Note that in this case $\mathbb{C}P^1$ has only three distinct orbits: \mathbb{C}^* , x_+ , and x_- . Thus, there are at most three different kinds of fibers for $\tilde{\varphi}$:

- (1) *generic fiber*: a connected component is given by the toric variety $\Sigma' \cap \varphi^{-1}(0)$; there are $[N : \varphi(N')]$ -many connected components.
- (2) *the fiber over x_+* : Since $N/N_{\sigma_+} = \{0\}$, by Lemma 2.1.11 the fiber can have only one connected component. The set of irreducible components of the fiber is given then by $\{Star(\sigma') \mid \sigma' \in \Sigma'_{\sigma_+}\}$.
- (3) *the fiber over x_-* : similar to (2) with the label + replaced by -.

For the dual picture, assume that $X_{\Sigma'}$ is projective and Σ' is the normal fan of a convex polytope Δ' in $M'_{\mathbb{R}}$. Then Δ' is decomposed into Δ'_{σ_+} , $\Delta'_{\{0\}}$, and Δ'_{σ_-} . The set of primitive faces in Δ'_{σ_+} is exactly the set of maximal faces of Δ'_{σ_+} . Similarly for Δ'_{σ_-} . The generic fiber of $\tilde{\varphi}$ corresponds to the projection of Δ' in $(M'/\varphi^{\dagger}(M))_{\mathbb{R}}$. For the fiber F_{x_+} over x_+ , since $\sigma_+^\perp = \{0\}$, each maximal face $\Theta_{\sigma'}$ in Δ'_{σ_+} as a polytope in σ'^{\perp} corresponds to an irreducible component of F_{x_+} . Similarly for the fiber F_{x_-} over x_- . (Cf. FIGURE 2-2-1.)

It is worth emphasizing that in this case the fiber over x_{\pm} is a union of codimension-1 orbit closures in $X_{\Sigma'}$; in other words, it is a T-Weil divisor in $X_{\Sigma'}$.

□

Example 2.2.2 [B-C-dlO-G]. In [B-C-dlO-G], Braun, Candelas, de la Ossa, and Grassi consider a 5-dimensional toric variety associated to a reflexive polytope, in which the Calabi-Yau hypersurfaces admit an elliptic fibration. In their work, they characterize the existence of the fibration and the generic fiber at the level of toric varieties and study also the gauge group of the effective 4-dimensional supersymmetric field theory and the divisors of the Calabi-Yau hypersurfaces that contribute to the superpotential of that effective field theory. Here let us study their example of toric fibration in more details. (The restriction of this fibration to Calabi-Yau hypersurfaces will be studied in detail in Example 4.2.1 in Sec. 4.2.) To make the comparison with their work easier, we shall follow their notations in describing the fan Σ' with modifications to fit into our earlier notations. The notations for the fan Σ are modified from [Ra].

(a) *The toric data of $X_{\Sigma'}$ and X_{Σ} .* The 1-skeleton of the fans and the labels of the 1-rays are given in the following table:

$\Sigma'(1)$		$\Sigma(1)$	
v'_1	(-1, 0, 2, 3)	d_4	(-1, 0, 0)
v'_2	(0,-1, 0, 2, 3)	d_3	(0,-1, 0)
c'_1	(0, 0,-1, 2, 3)	r_2	(0, 0,-1)
c'_2	(0, 0,-1, 1, 2)	r_1	(0, 0, 1)
v'_4	(0, 0, 0,-1, 0)	d_2	(0, 1, 2)
v'_5	(0, 0, 0, 0,-1)	u	(0, 1, 3)
b'	(0, 0, 0, 2, 3)	d_1	(1, 0, 4)
e'_1	(0, 0, 1, 2, 3)		
e'_2	(0, 0, 2, 2, 3)		
e'_3	(0, 0, 1, 1, 1)		
f'	(0, 1, 2, 2, 3)		
g'	(0, 1, 3, 2, 3)		
v'_6	(1, 0, 4, 2, 3)		

Denote a cone e.g. $[v'_1, b', e'_1, v'_2, v'_4]$, generated v'_1, b', e'_1, v'_2 , and v'_4 , by $v'_1 b' e'_1 v'_2 v'_4$ for brevity of notation. Then the 5-dimensional fan Σ' consists of 54 maximal (simplicial) cones :

$$\left\{ \begin{array}{l} v'_1 b' e'_1 v'_2 v'_4 , \quad v'_1 b' f' c'_1 v'_4 , \quad v'_1 b' v'_2 c'_1 v'_4 , \quad v'_1 e'_3 e'_1 v'_2 v'_5 , \quad v'_1 b' e'_1 v'_2 v'_5 , \quad v'_1 b' f' c'_1 v'_5 , \\ v'_1 b' v'_2 c'_1 v'_5 , \quad v'_1 e'_3 v'_2 v'_4 v'_5 , \quad v'_1 b' f' v'_4 g' , \quad v'_1 b' e'_1 v'_4 g' , \quad v'_1 b' f' v'_5 g' , \quad v'_1 e'_3 e'_1 v'_5 g' , \\ v'_1 b' e'_1 v'_5 g' , \quad v'_1 e'_3 v'_4 v'_5 g' , \quad v'_1 f' v'_4 v'_5 g' , \quad v'_1 e'_3 e'_1 v'_2 e'_2 , \quad v'_1 e'_3 v'_2 v'_4 e'_2 , \quad v'_1 e'_3 v'_2 v'_4 e'_2 , \\ v'_1 e'_3 e'_1 g' e'_2 , \quad v'_1 e'_3 v'_4 g' e'_2 , \quad v'_1 e'_4 v'_4 g' e'_2 , \quad v'_1 f' c'_1 v'_4 c'_2 , \quad v'_1 v'_2 c'_1 v'_4 c'_2 , \quad v'_1 f' c'_1 v'_5 c'_2 , \\ v'_1 v'_2 c'_1 v'_5 c'_2 , \quad v'_1 f' v'_4 v'_5 c'_2 , \quad v'_1 v'_2 v'_4 v'_5 c'_2 , \quad v'_6 b' e'_1 v'_2 v'_4 , \quad v'_6 b' f' c'_1 v'_4 , \quad v'_6 b' v'_2 c'_1 v'_4 , \\ v'_6 e'_3 e'_1 v'_2 v'_5 , \quad v'_6 b' e'_1 v'_2 v'_5 , \quad v'_6 b' f' c'_1 v'_5 , \quad v'_6 b' v'_2 c'_1 v'_5 , \quad v'_6 e'_3 v'_2 v'_4 v'_5 , \quad v'_6 b' f' v'_4 g' , \\ v'_6 b' e'_1 v'_4 g' , \quad v'_6 b' f' v'_5 g' , \quad v'_6 e'_3 e'_1 v'_5 g' , \quad v'_6 b' e'_1 v'_5 g' , \quad v'_6 e'_3 v'_4 v'_5 g' , \quad v'_6 b' f' v'_4 v'_5 g' , \\ v'_6 e'_3 e'_1 v'_2 c'_2 , \quad v'_6 e'_3 v'_2 v'_4 c'_2 , \quad v'_6 e'_1 v'_2 v'_4 e'_2 , \quad v'_6 e'_3 e'_1 g' e'_2 , \quad v'_6 e'_3 v'_4 g' e'_2 , \quad v'_6 e'_1 v'_4 g' e'_2 , \\ v'_6 f' c'_1 v'_4 c'_2 , \quad v'_6 v'_2 c'_1 v'_5 c'_2 , \quad v'_6 f' v'_4 v'_5 c'_2 , \quad v'_6 v'_2 c'_1 v'_5 c'_2 , \quad v'_6 f' v'_4 v'_5 c'_2 , \quad v'_6 v'_2 v'_4 v'_5 c'_2 , \end{array} \right\}$$

while the 3-dimensional fan Σ consists of 10 maximal (simplicial) cones :

$$\{ d_4 d_2 r_2 , d_4 d_2 u , d_4 u r_1 , d_4 d_3 r_1 , d_4 d_3 r_2 , d_1 d_2 r_2 , d_1 d_2 u , d_1 u r_1 , d_1 d_3 r_1 , d_1 d_3 r_2 \} .$$

Following [B-C-dlO-G] and denoting a fan as the summation of its maximal cones, Σ' can be decomposed into a combination of suspensions or extensions by extra rays :

$$\Sigma' = (v'_1 + v'_6)\Sigma'' \quad \text{and} \quad \Sigma'' = (f' + v'_2)\Sigma''^{(c)} + (g' + v'_2)\Sigma''^{(e)} + f'g'\Sigma''^{(0)},$$

where

$$\begin{aligned} \Sigma''^{(c)} &= b' c'_1 v'_4 + c'_1 c'_2 v'_4 + b' c'_1 v'_5 + c'_1 c'_2 v'_5 + c'_2 v'_4 v'_5 , \\ \Sigma''^{(e)} &= e'_1 e'_2 e'_3 + b' e'_1 v'_4 + e'_1 e'_2 v'_4 + e'_2 e'_3 v'_4 + b' e'_1 v'_5 + e'_1 e'_3 v'_5 + e'_3 v'_4 v'_5 , \\ \Sigma''^{(0)} &= b' v'_4 + b' v'_5 + v'_4 v'_5 . \end{aligned}$$

Observe that $\Sigma''^{(0)} = \Sigma''^{(c)} \cap \Sigma''^{(e)}$.

Similarly, Σ is the suspension by $\{d_4, d_1\}$ of the 2-dimensional subfan generated by $\Sigma(1) - \{d_4, d_1\}$ and can be written as

$$\Sigma = (d_4 + d_1)\Sigma^\sim, \quad \text{where} \quad \Sigma^\sim = (d_2 + d_3)r_2 + (u + d_3)r_1 + d_2u .$$

Note that all the cones in Σ have multiplicity 1 and hence X_Σ is a smooth toric 3-fold. It is the blow-up of the Hirzebruch 3-fold \mathbb{F}_{024} at a point, cf. Table 4.1 in [Ra].

(b) *The toric morphism $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_\Sigma$.* The map of fans $\varphi : \Sigma' \rightarrow \Sigma$ is induced by the projection map $\varphi : (y_1, y_2, y_3, y_4, y_5) \mapsto (y_1, y_2, y_3)$. Under this map,

$$\Sigma''^{(0)} \rightarrow 0, \quad \Sigma''^{(c)} - \Sigma''^{(0)} \rightarrow r_2, \quad \Sigma''^{(e)} - \Sigma''^{(0)} \rightarrow r_1.$$

Together with the correspondence

$$v'_1 \rightarrow d_4, \quad v'_6 \rightarrow d_1, \quad f' \rightarrow d_2, \quad g' \rightarrow u, \quad v'_2 \rightarrow d_3,$$

the set Σ'_σ of cones in Σ' associated to $\sigma \in \Sigma$ under φ and the set Σ'°_σ of primitive cones therein can be directly worked out. Furthermore, from Remark 2.1.12, $Ind(\sigma) = 1$ for all $\sigma \in \Sigma$ since φ is surjective. All these are listed in Columns 2 - 4 of Table 2-2-2 below.

$\sigma \in \Sigma$	Σ'_σ	$\tau' \in \Sigma'^\circ_\sigma$	$Ind(\sigma)$	the irreducible fiber component $F_\sigma^{\tau'}$
0	$\varphi^{-1}(0) = \Sigma'^{(0)}$	0	1	$WCP^2(1, 2, 3)$
d_4	$v'_1 \Sigma'^{(0)}$	v'_1	1	$WCP^2(1, 2, 3)$
d_3	$v'_2 \Sigma'^{(0)}$	v'_2	1	$WCP^2(1, 2, 3)$
r_2	$\Sigma'^{(c)} - \Sigma'^{(0)}$	c'_1, c'_2	1	$X(4), CP^2$
r_1	$\Sigma'^{(e)} - \Sigma'^{(0)}$	e'_1, e'_2, e'_3	1	$X(5), WCP^2(1, 1, 3), \mathbb{F}_2$
d_2	$f' \Sigma'^{(0)}$	f'	1	$WCP^2(1, 2, 3)$
u	$g' \Sigma'^{(0)}$	g'	1	$WCP^2(1, 2, 3)$
d_1	$v'_6 \Sigma'^{(0)}$	v'_6	1	$WCP^2(1, 2, 3)$
$d_2 r_2$	$f' (\Sigma'^{(c)} - \Sigma'^{(0)})$	$f' c'_1, f' c'_2$	1	$X(4), CP^2$
$d_2 u$	$f' g' \Sigma'^{(0)}$	$f' g'$	1	$WCP^2(1, 2, 3)$
$u r_1$	$g' (\Sigma'^{(e)} - \Sigma'^{(0)})$	$g' e'_1, g' e'_2, g' e'_3$	1	$X(5), WCP^2(1, 1, 3), \mathbb{F}_2$
$d_3 r_1$	$v'_2 (\Sigma'^{(e)} - \Sigma'^{(0)})$	$v'_2 e'_1, v'_2 e'_2, v'_2 e'_3$	1	$X(5), WCP^2(1, 1, 3), \mathbb{F}_2$
$d_3 r_2$	$v'_2 (\Sigma'^{(c)} - \Sigma'^{(0)})$	$v'_2 c'_1, v'_2 c'_2$	1	$X(4), CP^2$
$d_4 d_3$	$v'_1 v'_2 \Sigma'^{(0)}$	$v'_1 v'_2$	1	$WCP^2(1, 2, 3)$
$d_4 r_2$	$v'_1 (\Sigma'^{(c)} - \Sigma'^{(0)})$	$v'_1 c'_1, v'_1 c'_2$	1	$X(4), CP^2$
$d_4 r_1$	$v'_1 (\Sigma'^{(e)} - \Sigma'^{(0)})$	$v'_1 e'_1, v'_1 e'_2, v'_1 e'_3$	1	$X(5), WCP^2(1, 1, 3), \mathbb{F}_2$
$d_4 d_2$	$v'_1 f' \Sigma'^{(0)}$	$v'_1 f'$	1	$WCP^2(1, 2, 3)$
$d_4 u$	$v'_1 g' \Sigma'^{(0)}$	$v'_1 g'$	1	$WCP^2(1, 2, 3)$
$d_1 d_3$	$v'_6 v'_2 \Sigma'^{(0)}$	$v'_6 v'_2$	1	$WCP^2(1, 2, 3)$
$d_1 r_2$	$v'_6 (\Sigma'^{(c)} - \Sigma'^{(0)})$	$v'_6 c'_1, v'_6 c'_2$	1	$X(4), CP^2$
$d_1 r_1$	$v'_6 (\Sigma'^{(e)} - \Sigma'^{(0)})$	$v'_6 e'_1, v'_6 e'_2, v'_6 e'_3$	1	$X(5), WCP^2(1, 1, 3), \mathbb{F}_2$
$d_1 d_2$	$v'_6 f' \Sigma'^{(0)}$	$v'_6 f'$	1	$WCP^2(1, 2, 3)$
$d_1 u$	$v'_6 g' \Sigma'^{(0)}$	$v'_6 g'$	1	$WCP^2(1, 2, 3)$
$d_4 d_2 r_2$	$v'_1 f' (\Sigma'^{(c)} - \Sigma'^{(0)})$	$v'_1 f' c'_1, v'_1 f' c'_2$	1	$X(4), CP^2$
$d_4 d_2 u$	$v'_1 f' g' \Sigma'^{(0)}$	$v'_1 f' g'$	1	$WCP^2(1, 2, 3)$
$d_4 u r_1$	$v'_1 g' (\Sigma'^{(e)} - \Sigma'^{(0)})$	$v'_1 g' e'_1, v'_1 g' e'_2, v'_1 g' e'_3$	1	$X(5), WCP^2(1, 1, 3), \mathbb{F}_2$
$d_4 d_3 r_1$	$v'_1 v'_2 (\Sigma'^{(e)} - \Sigma'^{(0)})$	$v'_1 v'_2 e'_1, v'_1 v'_2 e'_2, v'_1 v'_2 e'_3$	1	$X(5), WCP^2(1, 1, 3), \mathbb{F}_2$
$d_4 d_3 r_2$	$v'_1 v'_2 (\Sigma'^{(c)} - \Sigma'^{(0)})$	$v'_1 v'_2 c'_1, v'_1 v'_2 c'_2$	1	$X(4), CP^2$
$d_1 d_2 r_2$	$v'_6 f' (\Sigma'^{(c)} - \Sigma'^{(0)})$	$v'_6 f' c'_1, v'_6 f' c'_2$	1	$X(4), CP^2$
$d_1 d_2 u$	$v'_6 f' g' \Sigma'^{(0)}$	$v'_6 f' g'$	1	$WCP^2(1, 2, 3)$
$d_1 u r_1$	$v'_6 g' (\Sigma'^{(e)} - \Sigma'^{(0)})$	$v'_6 g' e'_1, v'_6 g' e'_2, v'_6 g' e'_3$	1	$X(5), WCP^2(1, 1, 3), \mathbb{F}_2$
$d_1 d_3 r_1$	$v'_6 v'_2 (\Sigma'^{(e)} - \Sigma'^{(0)})$	$v'_6 v'_2 e'_1, v'_6 v'_2 e'_2, v'_6 v'_2 e'_3$	1	$X(5), WCP^2(1, 1, 3), \mathbb{F}_2$
$d_1 d_3 r_2$	$v'_6 v'_2 (\Sigma'^{(c)} - \Sigma'^{(0)})$	$v'_6 v'_2 c'_1, v'_6 v'_2 c'_2$	1	$X(4), CP^2$

TABLE 2-2-2. A toric description of the toric morphism $\tilde{\varphi}$. The irreducible components $F_\sigma^{\tau'}$ over the toric orbit O_σ is listed in the same order as τ' . The first four columns are explained in Part (a) and Part (b); the last column is explained in Part (c), in which the notations $X(4) = X_{\{(2,3),(-1,0),(-1,-1),(0,-1)\}}$ and $X(5) = X_{\{(2,3),(-1,0),(-2,-3),(-1,-2),(0,-1)\}}$ are used.

(c) *The fibers of $\tilde{\varphi}$.* From the table above, all the fibers of $\tilde{\varphi}$ are connected. Among the 33 torus orbits of X_Σ , the fiber is irreducible only over 15 of them. There are 59 primitive cones τ' in Σ' with respect to the map of fans φ . To see their relative

star $\text{Star}_\sigma(\tau')$, observe that

$$\begin{aligned}\Sigma''^{(c)} &= c'_1(b'v'_4 + c'_2v'_4 + b'v'_5 + c'_2v'_5) + (\text{cones without } c'_1) \\ &= c'_2(c'_1v'_4 + c'_1v'_5 + v'_4v'_5) + (\text{cones without } c'_2) \\ &= c'_1c'_2(v'_4 + v'_5) + (\text{cones without } c'_1c'_2).\end{aligned}$$

These identities together with that fact that all the cones explicitly shown are in Σ'_{r_2} give the combinatorial type of the fans for fibers $F_{r_2}, F_{d_2r_2}, F_{d_3r_2}, F_{d_4r_2}, F_{d_1r_2}, F_{d_4d_2r_2}, F_{d_4d_3r_2}, F_{d_1d_2r_2}, F_{d_1d_3r_2}$. They imply that each of these fibers has two irreducible components, which intersect at a toric-invariant \mathbb{CP}^1 . Observe also that

$$\begin{aligned}\Sigma''^{(e)} &= e'_1(e'_2e'_3 + b'v'_4 + e'_2v'_4 + b'v'_5 + e'_3v'_5) + (\text{cones without } e'_1) \\ &= e'_2(e'_1e'_3 + e'_1v'_4 + e'_3v'_4) + (\text{cones without } e'_2) \\ &= e'_3(e'_1e'_2 + e'_2v'_4 + e'_1v'_5 + v'_4v'_5) + (\text{cones without } e'_3) \\ &= e'_1e'_2(e'_3 + v'_4) + (\text{cones without } e'_1e'_2) \\ &= e'_2e'_3(e'_1 + v'_4) + (\text{cones without } e'_2e'_3) \\ &= e'_1e'_3(e'_2 + v'_5) + (\text{cones without } e'_1e'_3) \\ &= e'_1e'_2e'_3 + (\text{cones without } e'_1e'_2e'_3)\end{aligned}$$

These identities together with the fact that all the cones explicitly shown are in Σ'_{r_1} give the combinatorial type of the fans for fibers $F_{r_1}, F_{ur_1}, F_{d_3r_1}, F_{d_4r_1}, F_{d_1r_1}, F_{d_4ur_1}, F_{d_4d_3r_1}, F_{d_1ur_1}, F_{d_1d_3r_1}$. They imply also that each of these fibers has three irreducible components, any pair of which intersect at a toric-invariant \mathbb{CP}^1 and the three components intersect at a single point.

To determine $F_\sigma^{\tau'}$, one needs to know the lattice structure in $\varphi^{-1}((N_\sigma)_\mathbb{R})/(N'_{\tau'})_\mathbb{R}$ as well. It turns out that, in the current example, the lattice in $\varphi^{-1}((N_\sigma)_\mathbb{R})/(N'_{\tau'})_\mathbb{R}$ can be canonically identified with the lattice in $(\varphi^{-1}(0))_\mathbb{R}$ except for τ' that contain e'_2 . In the latter case, the lattice structure involved can be identified with the lattice $\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$ in $(\varphi^{-1}(0))_\mathbb{R}$ when an appropriate basis for the lattice $\varphi^{-1}(0)$ is chosen. One can also check that $F_\sigma^{\tau'}$ depends only on which or none of $c'_1, c'_2, e'_1, e'_2, e'_3$ is contained in τ' .

All these observations together with some explicit calculations give us a complete description of all the irreducible components $F_\sigma^{\tau'}$ of F_σ and their intersections in F_σ , as listed in the last column of the Table 2-2-2 and indicated in FIGURE 2-2-3 via fans. In particular, there are 6 different kinds of toric varieties that an irreducible fiber components $F_\sigma^{\tau'}$ can assume.

ray involved	•	c'_1	c'_2	e'_1	e'_2	e'_3
toric variety	$\text{WCP}^2(1, 2, 3)$	$X(4)$	\mathbb{CP}^2	$X(5)$	$\text{WCP}^2(1, 1, 3)$	\mathbb{F}_2
reflexivity	yes	yes	yes	no	no	yes

where $\text{WCP}^2(1, 2, 3)$ and $\text{WCP}^2(1, 1, 3)$ are weighted projective spaces, $X(4)$ (resp.

$X(5, \mathbb{F}_2)$ is the toric variety associated to the 2-dimensional fan determined by the set of 1-cones $\{(2, 3), (-1, 0), (-1, -1), (0, -1)\}$ (resp. $\{(2, 3), (-1, 0), (-2, -3), (-1, -2), (0, -1)\}, \{(1, 0), (0, 1), (-1, 2), (0, -1)\}$).

Figure 1: FIGURE 2-2-3. The fans for the generic fiber and the reducible nongeneric fibers are indicated. An reducible toric variety can be coded in a multi-fan; the intersection relations of irreducible components can also be read off from it. The 1-cones in $\Sigma'(1)$ that give rise to the corresponding cones for the fiber are used as labels. The T-Weil divisor D_i associated to each ray is also indicated for later use. (For clarity of picture, all the "'''s are omitted.)

□

Remark 2.2.3. Note that in Example 2.2.2, each non-degenerate fiber contains an irreducible component that is associated to a reflexive polytope (while it may have other components that are not). It is interesting to know if this is always so for toric morphisms from fans related to reflexive polytopes. If not, then how to characterize those fibrations that have this property.

3 Induced morphism and fibers for hypersurfaces.

Having understood the fibers of a toric morphism $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma}$, our next theme is the study of the induced morphism $\tilde{\varphi}|_{Y'} : Y' \rightarrow X_{\Sigma}$ for a hypersurface Y' in $X_{\Sigma'}$. Since hypersurfaces in $X_{\Sigma'}$ that we will be interested in are realized as the zero locus of sections of some line bundle over $X_{\Sigma'}$, the problem can be converted to the study of the various restrictions of a line bundle over $X_{\Sigma'}$ with a section. The general study of this is given in this section. The results presented here will then be applied to the case of Calabi-Yau hypersurfaces in Sec. 4.

3.1 Preparations.

We give first some basic lemmas relating $\tilde{\varphi}$ and sections of a line bundle \mathcal{L} over $X_{\Sigma'}$ and then use the flattening stratification of $\tilde{\varphi}$ described in Proposition 2.1.4 in Sec. 2.1 to study $\tilde{\varphi}|_{Y'}$ for Y' realized as the zero-locus of a section of \mathcal{L} .

Recall first the following facts: (Proposition 2.4 and Corollary 2.5 in [Od2])

Fact 3.1.1 [equivariant line bundle]. *Let $X' = X_{\Sigma'}$, $SF(\Sigma')$ be the group of continuous functions from $N'_{\mathbb{R}}$ to \mathbb{R} that are linear on each $\sigma' \in \Sigma'$, and $CDiv_{T_{N'}}(X')$ be the group of $T_{N'}$ -invariant Cartier divisors on X' . For $h' \in SF(\Sigma')$, let $D_{h'}$ be the Cartier divisor on X' associated to h' . Then*

- (1) *Given a $T_{N'}$ -invariant Cartier divisor D' on X' , then the restriction D' to each affine open set $U_{\sigma'}$, $\sigma' \in \Sigma'$, coincides with a principal divisor on $U_{\sigma'}$ so that $D' = D_{h'}$ for some $h' \in SF(\Sigma')$. The map $h' \mapsto D_{h'}$ gives an isomorphism $SF(\Sigma') \xrightarrow{\sim} CDiv_{T_{N'}}(X')$.*
- (2) *For any Cartier divisor D' on X' , one has an $\mathcal{O}_{X'}$ -module isomorphism $\mathcal{O}_{X'}(D') \simeq \mathcal{O}_{X'}(D_{h'})$ for some $h' \in SF(\Sigma')$. Thus, the composition of natural homomorphisms $SF(\Sigma') \rightarrow CDiv(X') \rightarrow \text{Pic}(X')$ is surjective.*
- (3) *The following are equivalent for $h' \in SF(\Sigma')$:*
 - (a) $h' \in M'$.
 - (b) $D_{h'}$ is a principal divisor.
 - (c) $\mathcal{O}_{X'}(D_{h'}) \simeq \mathcal{O}_{X'}$ as $\mathcal{O}_{X'}$ -modules.
- (4) *As a corollary of (1), (2), and (3), one has the canonical isomorphism*

$$\begin{aligned} SF(\Sigma')/M' &\xrightarrow{\sim} CDiv_{T_{N'}}(X')/CDiv_{T_{N'}}(X') \cap PDiv(X') \\ &\xrightarrow{\sim} ELB(X')/\{\mathcal{O}_{X'}(m') \mid m' \in M'\} \xrightarrow{\sim} \text{Pic}(X'), \end{aligned}$$

where $PDiv(X')$ is the group of principal divisors on X' , $ELB(X')$ is the group of equivariant line bundles over X' , and $\mathcal{O}_{X'}(m')$ is the equivariant trivial line bundle $\mathcal{O}_{X'}$ with the linearization given by the character $\chi^{m'}$.

Remark 3.1.2 ([Fu] and [Od2], also [Ka]). An element $h' \in SF(\Sigma')$ determines (non-uniquely) a system of weights $W_{h'} = \{m'_{\sigma'} \in M' \mid \sigma' \in \Sigma'\}$, from which the following data are specified:

- (1) The linearization for the trivialization of $\mathcal{L}_{h'}$ when restricted to the affine charts $U_{\sigma'}$ for $X_{\Sigma'}$. Explicitly, this is given by $t' \cdot (x', a) = (t' \cdot x', \chi^{m'_{\sigma'}}(t')a)$, where $t' \in T_{N'}$ and $(x', a) \in U_{\sigma'} \times \mathbb{C}$.
- (2) The transition function between local trivializations: $f_{\sigma'_1 \sigma'_2}: U_{\sigma'_1} \cap U_{\sigma'_2} \rightarrow \mathbb{C}^{\times}$ from $\mathcal{L}_{h'}|_{U_{\sigma'_1} \cap U_{\sigma'_2} \subset U_{\sigma'_1}}$ to $\mathcal{L}_{h'}|_{U_{\sigma'_1} \cap U_{\sigma'_2} \subset U_{\sigma'_2}}$ by $\chi^{m'_{\sigma'_2} - m'_{\sigma'_1}}$.
- (3) A $T_{N'}$ -equivariant meromorphic section, namely $\{\chi^{m'_{\sigma'}}\}_{\sigma' \in \Sigma'}$, in $\mathcal{L}_{h'}$.

The weights $\{m'_{\sigma'} \in M' \mid \sigma' \in \Sigma'(n')\}$ determines an integral polytope $\Delta_{h'}$ in $M'_{\mathbb{R}}$ by taking the convex hull and one has

$$\begin{array}{ccc} \oplus_{m' \in \Delta_{h'} \cap M'} \mathbb{C} \cdot \chi^{m'} & \simeq & H^0(X_{\Sigma'}, \mathcal{L}_{h'}) \\ m' & \mapsto & s_{m'} \end{array},$$

where $s_{m'} = \chi^{m' - m'_{\sigma'}}$ over each affine chart $U_{\sigma'}$, $\sigma' \in \Sigma'$. The compatibility of elements in $W_{h'}$ implies that Σ' is a refinement of the normal fan of $\Delta_{h'}$.

Remark 3.1.3 [$T_{N'}$ -action on $H^0(X_{\Sigma'}, \mathcal{O}_{X'}(D_{h'}))$]. The $T_{N'}$ -action on $X_{\Sigma'}$ together with the linearization of $\mathcal{O}_{X'}(D_{h'})$ determined by the weight system $W_{h'}$ in Remark 3.1.2 determines a $T_{N'}$ -action on $H^0(X_{\Sigma'}, \mathcal{O}_{X'}(D_{h'}))$. Let $t' \in T_{N'}$; then Remark 3.1.2 implies that $t' \cdot s_{m'} = \chi^{-m'}(t') \cdot s_{m'}$ over $U_{\sigma'}$ for $m' \in \Delta' \cap M'$.

Fact 3.1.4 [restriction of line bundle to orbit closure]. (Cf. [Fu].) *Given a line bundle $\mathcal{L}_{\Delta'}$ over $X_{\Sigma'}$. Let $h' \in SF(\Sigma')$ be the associated continuous piecewise linear function on $N'_{\mathbb{R}}$ with respect to Σ' . Then the restriction of $\mathcal{L}_{\Delta'}$ to an orbit closure $V(\tau')$ can be described as follows:*

- (1) *Let $m'_{\tau'} \in M'$ be a linear function on $N'_{\mathbb{R}}$ such that $m'_{\tau'}|_{\tau'} = h'|_{\tau'}$. Then $h'' = h' - m'_{\tau'} \in SF(\Sigma')$ defines a line bundle $\mathcal{L}_{h''}$ isomorphic to $\mathcal{L}_{h'}$ (with different linearization). $\mathcal{L}_{h''}$ corresponds to the polytope $\Delta' - m'_{\tau'}$.*
- (2) *After this shift, the restriction of h'' to the set of cones σ' with $\tau' \prec \sigma'$ descends to a continuous piecewise linear function on $(N'/N'_{\sigma'})_{\mathbb{R}}$ with respect to $\text{Star}(\tau')$. This defines a line bundle over $V(\tau')$ isomorphic to $\mathcal{L}_{\Delta'}|_{V(\tau')}$ (with perhaps different linearization).*
- (3) *In terms of polytopes, let σ'_i , $i = 1, \dots, k$, be the maximal cones in Σ' with $\tau' \prec \sigma'_i$. Let m'_i be the vertices of Δ' corresponding to the restriction $h'|_{\sigma'_i}$. Since all m'_i give the same restriction to τ' , all the differences $m'_i - m'_j$ lie in sublattice $\tau'^{\perp} \cap M'$. The convex hull $\Delta'_{\tau'}$ of $\{m'_1, \dots, m'_k\}$ in $M'_{\mathbb{R}}$ is then contained in, say, $m'_1 + \tau'^{\perp}$ and hence determines an integral polytope in τ'^{\perp} , unique up to a translation by a lattice point. This is the polytope in τ'^{\perp} that gives $\mathcal{L}_{\Delta'}|_{V(\tau')}$, up to a linearization.*

The following lemma characterizes $\Delta'_{\tau'}$ up to a translation.

Lemma 3.1.5. $\Delta'_{\tau'} = (\Delta' - m'_{\tau'}) \cap \tau'^{\perp}$ up to an integral translation.

Proof. After the translation, $\Delta'_{\tau'}$ becomes the convex hull of $\{m'_1 - m'_{\tau'}, \dots, m'_k - m'_{\tau'}\}$ contained in $(\Delta' - m'_{\tau'}) \cap \tau'^{\perp}$. If there is an extra integral point m'_0 in $(\Delta' - m'_{\tau'}) \cap \tau'^{\perp}$ besides those in $\Delta'_{\tau'}$, then, since $m'_0 \in (\Delta' - m'_{\tau'})$, it gives rise to a holomorphic section in $\mathcal{L} = \mathcal{L}_{\Delta' - m'_{\tau'}}$ and hence on $\mathcal{L}|_{V(\tau')}$. On the other hand, since

it is outside $\Delta'_{\tau'}$, it cannot be holomorphic on $V(\tau')$. This leads to a contradiction and one concludes the lemma.

□

Consider now the restriction homomorphism

$$\text{res} : H^0(X_{\Sigma'}, \mathcal{L}_{\Delta'}) \longrightarrow H^0(V(\tau'), \mathcal{L}_{\Delta'_{\tau'}})$$

that sends a holomorphic section of $\mathcal{L}_{\Delta'}$ to its restriction over $V(\sigma')$. Note that for $m' \in (\Delta' \cap M') - \Delta'_{\tau'}$, the associated holomorphic section in $\mathcal{L}_{\Delta'}$ restricted to $V(\tau')$ gives the 0-section in $\mathcal{L}_{\Delta'_{\tau'}}$. The lemma below follows immediately from Remark 3.1.2:

Lemma 3.1.6 [sections restricted to orbit closure]. *The restriction homomorphism $\text{res} : H^0(X_{\Sigma'}, \mathcal{L}_{\Delta'}) \rightarrow H^0(V(\sigma'), \mathcal{L}_{\Delta'_{\tau'}})$ is surjective with kernel generated by the sections in $\mathcal{L}_{\Delta'}$ associated to $\Delta' \cap M' - \Delta'_{\tau'}$. The restriction to $V(\tau')$ of the section $s_{m'}$ in $\mathcal{L}_{\Delta'}$ associated to $m' \in \Delta'_{\tau'} \cap M' \subset \Delta'$ is the section in $\mathcal{L}_{\Delta'_{\tau'}}$ given by the same $m' \in \Delta'_{\tau'}$.*

Finally, let us turn to the restriction of a line bundle to a generic fiber of a toric morphism. The following lemma characterizes pullback line bundles and sections from a toric morphism in terms of toric data:

Lemma 3.1.7 [pullback of line bundle and section]. *Given lattices N_1, N_2 , and fans Σ_i in $N_i\mathbb{R}$, $i = 1, 2$. Let $\tilde{f} : X_{\Sigma_1} \rightarrow X_{\Sigma_2}$ be a toric morphism associated to a map of fans $f : \Sigma_1 \rightarrow \Sigma_2$. Let M_i be the dual lattice of N_i and $f^\dagger : M_2 \rightarrow M_1$ be the dual of f . Suppose that Δ_2 is an integral polytope in $M_2\mathbb{R}$. Then $\tilde{f}^*\mathcal{L}_{\Delta_2} = \mathcal{L}_{f^\dagger(\Delta_2)}$. Furthermore, if s is a section of \mathcal{L}_{Δ_2} that corresponds to a lattice point $m \in \Delta_2$, then f^*s is the section of $\tilde{f}^*\mathcal{L}_{\Delta_2}$ that corresponds to $f^\dagger(m) \in f^\dagger(\Delta_2)$.*

Proof. Let $h_2 \in SF(\Sigma_2)$ be the piecewise linear function on $N_2\mathbb{R}$ such that $\mathcal{L}_{h_2} = \mathcal{L}_{\Delta_2}$. Then the first assertion follows from the isomorphisms $\tilde{f}^*\mathcal{L}_{\Delta_2} = \tilde{f}^*\mathcal{L}_{h_2} = \mathcal{L}_{h_2 \circ f} = \mathcal{L}_{f^\dagger(\Delta_2)}$ as equivariant line bundles.

For the second assertion, recall from [Od2] that the piecewise linear function h_2 determines a local linearized trivialization of \mathcal{L}_{Δ_2} over the affine chart U_{σ_2} for each $\sigma_2 \in \Sigma_2$ together with a meromorphic section up to a constant multiple. This meromorphic section over U_{σ_2} is determined by the restriction $h_2|_{\sigma_2}$, which can be represented by an $m_{\sigma_2} \in \Delta_2 \cap M_2$. The section s associated to m_2 is then realized by the character $\chi^{m_2 - m_{\sigma_2}}$ over U_{σ_2} . Note also that, for $\sigma'_1 \in \Sigma_1$, let $m_{\sigma'_1} = f^\dagger(m_{\sigma'_2})$, where σ'_2 is the unique cone in Σ_2 such that the interior of σ'_1 is mapped by f to the interior of σ'_2 , then $f^\dagger(\Delta_2)$ is the convex hull of $\{m_{\sigma'_1} \mid \sigma'_1 \in \Sigma_1\}$.

Now suppose that the interior of $\sigma_1 \in \Sigma_1$ is mapped by f to the interior of σ_2 . Then $f(U_{\sigma_1}) \subset U_{\sigma_2}$ and $\tilde{f}^*(s|_{U_{\sigma_2}})$ is a holomorphic section in $\tilde{f}^*(\mathcal{L}_{\Delta_2}|_{U_{\sigma_2}})$

represented by the character $\chi^{f^\dagger(m_2) - f^\dagger(m_{\sigma_2})}$. Since the local meromorphic section of $\tilde{f}^*(\mathcal{L}_{\Delta_2})|_{U_{\sigma_1}}$ determined by $f^\dagger(\Delta_2)$ is given by $\chi^{f^\dagger(m_{\sigma_2})}$ and this discussion holds for every $\sigma_2 \in \Sigma_2$ that contains some cone in $f(\Sigma_1)$, $f^\dagger(m_2)$ must correspond to \tilde{f}^*s . This concludes the proof.

□

Recall the toric morphism $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_\Sigma$ and the notations from Sec. 2.

Corollary 3.1.8 [line bundle and sections restricted to fiber]. *Let $\mathcal{L}_{\Delta'}$ be a line bundle over $X_{\Sigma'}$.*

- (a) *The restriction of $\mathcal{L}_{\Delta'}$ to the irreducible component $F_\sigma^{\tau'}$ of a fiber is given by the polytope $\overline{\Delta_{\tau'}^\perp}$ in $\tau'^\perp/\varphi^\dagger(\sigma^\perp)$. In particular, the (non-linearized) isomorphism class of the restriction of $\mathcal{L}_{\Delta'}$ to each fiber over the orbit O_σ in X_Σ depends only on σ .*
- (b) *The restriction to $\mathcal{L}_{\Delta'}|_{F_\sigma^{\tau'}}$ of a section in $\mathcal{L}_{\Delta'}$ that corresponds to $m' \in \Delta_{\tau'} \cap M'$ is a constant multiple of the section in $\mathcal{L}_{\Delta'}|_{F_\sigma^{\tau'}}$ that corresponds to the lattice point $\overline{\varphi^\dagger}_\sigma(m') \in \overline{\Delta_{\tau'}}$.*

FIGURE 3-1-1.

Proof. Since $F_\sigma^{\tau'}$ is the generic fiber of the restriction of $\tilde{\varphi}$ to the toric variety $V(\tau')$, without loss of generality, we only need to prove both assertions for the generic fiber of $\tilde{\varphi}$. In this case, consider the map of fans $\varphi^{-1}(0) \rightarrow \Sigma'$ induced by the inclusion map $\tilde{\varphi}^{-1}(0) \hookrightarrow N'_\mathbb{R}$. Then both assertions follow from Lemma 3.1.7.

□

Remark 3.1.9. When Σ' is the normal fan of Δ' , in which case h' is strictly convex, $\Delta'_{\tau'} = \Theta_{\tau'}$, the facet of Δ' dual to τ' .

3.2 The induced morphism: $Y' \subset X_{\Sigma'} \rightarrow X_\Sigma$.

Let Y' be a hypersurface of $X_{\Sigma'}$ realized as the zero-locus of a section of $\mathcal{L}_{\Delta'}$ and recall the flattening stratification for the toric morphism $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_\Sigma$, discussed in Sec. 2.1, Proposition 2.1.4. Our strategy of understanding the induced morphism $\varphi|_{Y'} : Y' \rightarrow X_\Sigma$ is to understand the restriction of the flattening stratification of $\tilde{\varphi}$ to $\tilde{\varphi}|_{Y'}$. A flattening stratification of $\tilde{\varphi}|_{Y'}$, with each stratum a topological bundle is then obtained by a refinement of this restriction. We discuss first a fundamental case and then all other cases, which can be reduced to the fundamental case. The notations here follow those in Sec. 2.

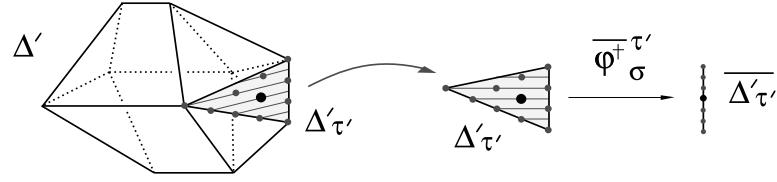


FIGURE 3-1-1. Restriction of a line bundle $\mathcal{L}_{\Delta'}$ with a section $s_{m'}$ to an irreducible fiber $F_{\sigma}^{\tau'}$ can be understood in two steps: (1) restriction to $V(\tau')$, which is described by $\Delta'_{\tau'}$; and (2) further restriction to the generic fiber $F_{\sigma}^{\tau'}$ of the toric morphism $\tilde{\varphi} : V(\tau') \rightarrow V(\sigma)$ from the restriction of $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma}$. The latter is described by the quotient $\overline{\varphi^+}_{\sigma}^{\tau'}(m') \in \overline{\Delta'_{\tau'}}$. (m' and its image are indicated by the thickened darkest lattice points.)

Case (1) [fundamental]: Restriction to generic fibers for φ surjective.

Let $\xi : N \rightarrow N'$ be a homomorphism such that $\varphi \circ \xi = Id_N$. Then ξ determines a splitting of N' by

$$\begin{aligned} N' &= \ker \varphi + \text{Im } \xi & \simeq & \ker \varphi \oplus N \\ v &= (v - \xi \circ \varphi(v)) + \xi \circ \varphi(v) & \mapsto & (v - \xi \circ \varphi(v), \varphi(v)). \end{aligned}$$

On the dual side, recall the quotient map $\overline{\varphi^+} : M' \rightarrow M'/\varphi^+(M)$, then $M'/\varphi^+(M)$ is canonically isomorphic to $\ker \xi^{\dagger}$. Thus, there is a map $i_{\xi} : M'/\varphi^+(M) \rightarrow M'$ such that $\xi^{\dagger} \circ i_{\xi} = 0$ and that $\overline{\varphi^+} \circ i_{\xi} = Id_{M'/\varphi^+(M)}$. From this, one obtains a decomposition:

$$\begin{aligned} M' &= \ker \xi^{\dagger} + \text{Im } \varphi^{\dagger} & \simeq & M'/\varphi^+(M) \oplus M \\ m' &= i_{\xi} \circ \widetilde{\varphi^+}(m') + (m' - i_{\xi} \circ \widetilde{\varphi^+}(m')) & \mapsto & (\overline{\varphi^+}(m'), \xi^{\dagger}(m')). \end{aligned}$$

Note that different choices of ξ give different projections to the second component of the decomposition $(\overline{\varphi^+}, \xi^{\dagger}) : M' \xrightarrow{\sim} M'/\varphi^+(M) \oplus M$.

Let Δ' be a rational convex polytope in $M'_{\mathbb{R}}$ and $\mathcal{L}_{\Delta'}$ be the associated line bundle over $X_{\Sigma'}$. Note that, since ξ in general does not induce a map of fans from Σ to Σ' , the polytope $\xi^{\dagger}(\Delta')$ in $M_{\mathbb{R}}$ may not give a line bundle over X_{Σ} . However, $\Sigma_{\xi} = \tilde{\varphi}(\xi(N_{\mathbb{R}}) \cap \Sigma')$ is a fan in $N_{\mathbb{R}}$ that refines Σ . Thus $\xi^{\dagger}(\Delta')$ corresponds to the line bundle $\xi^* \mathcal{L}_{\Delta'}$ over $X_{\Sigma_{\xi}}$. After pushing forward by the morphism $X_{\Sigma_{\xi}} \rightarrow X_{\Sigma}$, we shall regard $\widetilde{\xi}^* \mathcal{L}_{\Delta'}$ as a rank-one torsion-free sheaf over X_{Σ} .

Lemma 3.2.1 [factorization over orbit]. *Let $O_0 = T_N$ be the T_N -orbit in X_{Σ} corresponding to $\{0\} \in \Sigma$ and F_0 be the fiber of $\tilde{\varphi}$ over O_0 . Then:*

(a) ξ determines a product decomposition $\tilde{\varphi}^{-1}(O_0) = O_0 \times F_0$. Let pr_2 be the corresponding projection map to the second factor, then $\mathcal{L}_{\Delta'}|_{\tilde{\varphi}^{-1}(O_0)} \simeq pr_2^* \mathcal{L}_{\overline{\Delta'}}$.

(b) The surjective maps $\xi^\dagger(\Delta') \xleftarrow{\xi^\dagger} \Delta' \xrightarrow{\overline{\varphi^\dagger}} \overline{\Delta'}$ induce a factorization

$$\begin{aligned} H^0(\tilde{\varphi}^{-1}(O_0), \mathcal{L}_{\Delta'}|_{\tilde{\varphi}^{-1}(O_0)}) &\hookrightarrow H^0(O_0, \mathcal{O}_{O_0}) \otimes H^0(F_0, \mathcal{L}_{\overline{\Delta'}}) \\ s_{m'} &\mapsto s_{\xi^\dagger(m')} \otimes s_{\overline{\varphi^\dagger}(m')} , \end{aligned}$$

where $m' \in \Delta'$. (Here we are identifying \mathcal{O}_{O_0} with $\mathcal{L}_{\xi^\dagger(\Delta')}|_{O_0}$ implicitly.)

(c) Let $\xi_1, \xi_2 : N \rightarrow N'$ be two homomorphisms such that $\varphi \circ \xi_i = Id_N$, $i = 1, 2$. Then, the $H^0(O_0, \mathcal{O}_{O_0})$ -factor of the factorizations in Part (b) with respect to ξ_1 and ξ_2 respectively are related by $s_{\xi_2^\dagger(m')} = \chi^{(\xi_2^\dagger - \xi_1^\dagger)(m')} s_{\xi_1^\dagger(m')}$ while the $H^0(F_0, \mathcal{L}_{\overline{\Delta'}})$ -factors are the same.

Proof. As a toric variety, $\tilde{\varphi}^{-1}(O_0)$ is described by the fan $\Sigma'_{\{0\}}$ in $N'_\mathbb{R}$ while F_0 is described by the same fan but in the subspace $\varphi_\mathbb{R}^{-1}(0)$ in $N'_\mathbb{R}$. The map ξ induces a projection map $\pi_\xi : \tilde{\varphi}^{-1}(O_0) \rightarrow F_0$ that projects the affine chart $U_{\sigma'}$ of $\tilde{\varphi}^{-1}(O_0)$ to the affine chart $\overline{U}_{\sigma'}$ of F_0 for $\sigma' \in \Sigma'_{\{0\}}$. It also induces a T_N -action on $\tilde{\varphi}^{-1}(O_0)$ that descends under π_ξ to the identity group action on F_0 . Together, this specifies a decomposition $\tilde{\varphi}^{-1}(O_0) = O_0 \times F_0$. Let $h' \in SF(\Sigma')$ be a piecewise linear function on $N'_\mathbb{R}$ that determines $\mathcal{L}_{\Delta'}$. Then $\mathcal{L}_{\Delta'}|_{\tilde{\varphi}^{-1}(O_0)}$ is determined by $h'|_{\varphi_\mathbb{R}^{-1}(0)}$, whose corresponding polytope in M' can be chosen to be $i_\xi(\overline{\Delta'})$. This implies that $\mathcal{L}_{\Delta'}|_{\tilde{\varphi}^{-1}(O_0)} \simeq pr_2^* \mathcal{L}_{\overline{\Delta'}}$ and, hence, proves Part (a).

For Part (b), first observe that the set of affine charts $\mathcal{U} = \{U_{\sigma'} | \sigma' \in \Sigma'_{\{0\}}\}$ forms a covering of $\tilde{\varphi}^{-1}(O_0)$. Over the affine chart $U_{\sigma'}$, $\sigma' \in \Sigma'_{\{0\}}$, of $\tilde{\varphi}^{-1}(O_0)$, one can decompose the section $s_{m'}$ in $\mathcal{L}_{\Delta'}|_{\tilde{\varphi}^{-1}(O_0)}$ associated to $m' \in \Delta' \cap M'$ as

$$s_{m'}|_{U_{\sigma'}} = \chi^{m' - m'_{\sigma'}} = \chi^{\xi^\dagger(m') - \xi^\dagger(m'_{\sigma'})} \cdot \chi^{\overline{\varphi^\dagger}(m') - \overline{\varphi^\dagger}(m'_{\sigma'})} = \chi^{\xi^\dagger(m')} \cdot \chi^{\overline{\varphi^\dagger}(m') - \overline{\varphi^\dagger}(m'_{\sigma'})}$$

since $\xi^\dagger \circ i_\xi = 0$. One notes that $\chi^{\xi^\dagger(m')}$ is the section $s_{\xi^\dagger(m')}$ of \mathcal{O}_{O_0} and $\chi^{\overline{\varphi^\dagger}(m') - \overline{\varphi^\dagger}(m'_{\sigma'})}$ is the section $s_{\overline{\varphi^\dagger}(m')}$ of $\mathcal{L}_{\overline{\Delta'}}$ restricted to $\overline{U}_{\sigma'}$. Since the transition map between the affine charts of $\tilde{\varphi}^{-1}(O_0)$ in \mathcal{U} is trivial along the O_0 -component of the product decomposition of $\tilde{\varphi}^{-1}(O_0)$, this proves Part (b).

Finally, Part (c) follows from the proof of Part (b). This concludes the proof. \square

Let $\xi_1, \xi_2 : N \rightarrow N'$ be two homomorphisms such that $\varphi \circ \xi_i = Id_N$, $i = 1, 2$. Then the difference $\xi_2 - \xi_1$ induces morphisms

$$\begin{array}{ccc} N & \xrightarrow{\xi_{21}} & \varphi^{-1}(0) \\ M'/\varphi^\dagger(M) & \xrightarrow{\xi_{21}^\dagger} & M \\ T_N & \xrightarrow{\xi_{21}^T} & T_{\varphi^{-1}(0)} \\ \{\text{character of } T_{\varphi^{-1}(0)}\} & \xrightarrow{\xi_{21}^X} & \{\text{character of } T_N\} \end{array} .$$

In terms of these, the transition factor $\chi^{(\xi_2^\dagger - \xi_1^\dagger)(m')}$ in Part (c) of Lemma 3.2.1, as a character of T_N , is simply $\xi_{21}^X(\chi^{\overline{\varphi^\dagger}(m')}) = \chi^{\overline{\varphi^\dagger}(m')} \circ \xi_{21}^T$. Thus, Lemma 3.2.1 and Remark 3.1.3 together imply :

Corollary 3.2.2 [torically equivalent restriction]. *The description of the restriction of a section of $\mathcal{L}_{\Delta'}$ to a fiber F_0 over O_0 by different choices of ξ give rise to torically equivalent sections in $H^0(F_0, \mathcal{L}_{\overline{\Delta'}})$.*

In particular, when one considers the hypersurfaces from the zero-locus of sections of $\mathcal{L}_{\Delta'}$, though a ξ has to be chosen to give an explicit description of its restriction to the generic fiber, the choice of ξ does not change the toric equivalence class of the hypersurfaces in the fibers.

Remark 3.2.3. The situation here is pretty much like local trivialization of a bundle: There is no canonical one; however, different local trivializations are related by automorphisms of the fiber.

Case (2): Restriction to exceptional fibers for φ surjective.

Recall from Remark 2.1.12 that, in this case, $Ind(\sigma) = [N/N_\sigma : \overline{\varphi}_\sigma^{\sigma'}(N'/N'_{\sigma'})] = 1$ for any $\sigma \in \Sigma$ and $\sigma' \in \Sigma'_\sigma$. Hence, $\overline{\varphi}_\sigma^{\sigma'} : N'/N'_{\sigma'} \rightarrow N/N_\sigma$ is surjective and $\widetilde{\varphi}_\sigma^{\sigma'^{-1}}(O_\sigma) = \widetilde{\varphi}^{-1}(O_\sigma) = O_\sigma \times F_\sigma^c$. Let τ' be a primitive element in Σ'_σ , then the orbit closure $V(\tau')$ of τ' in $X_{\Sigma'}$ is an irreducible component of the closure of $\widetilde{\varphi}^{-1}(O_\sigma)$ in $X_{\Sigma'}$ and $F_\sigma^{\tau'}$ is a generic fiber of the restriction $\widetilde{\varphi}|_{V(\tau')} : V(\tau') \rightarrow V(\sigma)$. Consequently, for each pair (σ, τ') , where $\sigma \in \Sigma$ and $\tau' \in \Sigma'_\sigma$ primitive, one can apply the discussion in Preparations to obtain the restriction of $\mathcal{L}_{\Delta'}$ and the sections therein to $V(\tau')$ and then the discussions in Case (1) [fundamental] to this bundle over $V(\tau')$. In this way, one can write down how the generic fiber of the morphism $Y' \cap V(\tau') \rightarrow V(\sigma)$ varies along $O(\sigma)$.

Since each irreducible component of a fiber of $\widetilde{\varphi}$ must appear as a generic fiber for some (σ, τ') , the discussions in Case (1) and above together gives us a description of every fiber of $Y' \rightarrow X_\Sigma$ orbit by orbit.

Together this concludes the discussion for the case $\varphi : N' \rightarrow N$ surjective.

Case (3) : Restriction to fibers when the index $[N : \varphi(N')]$ is finite.

The maps $N' \rightarrow \varphi(N') \hookrightarrow N$ induce a decomposition of $\tilde{\varphi} : X_{\Sigma'} \xrightarrow{\tilde{\varphi}_1} X_{\Sigma} \xrightarrow{\tilde{\iota}_1} X_{\Sigma}$, where the first Σ is a fan in $\varphi(N')_{\mathbb{R}}$ while the second Σ is in $N_{\mathbb{R}}$. The induced fibration $\tilde{\varphi}_1 : Y' \rightarrow X_{\Sigma}$ is already discussed in Cases (1) and (2) above. Thus, it remains to understand the map $\tilde{\iota}_1$, which is a branched covering of order $[N : \varphi(N')]$ and is a special case of our discussion in Sec. 2.1. The number of sheets over O_{σ} , $\sigma \in \Sigma$, is the same as $Ind(\sigma)$, which must satisfy $Ind(\sigma) \leq [N : \varphi(N')]$. The branched locus $Br(\tilde{\varphi})$ of $\tilde{\varphi}$ is a union of $V(\sigma)$ with $Ind(\sigma) < [N : \varphi(N')]$. The fiber of $\tilde{\varphi} : Y' \rightarrow X_{\Sigma}$ over a point p is then the disjoint union $\cup_{p_i \in \tilde{\iota}_1^{-1}(p)} \tilde{\varphi}_1|_{Y'}^{-1}(p_i)$. This concludes the discussion.

Case (4) : Restriction to fibers when the index $[N : \varphi(N')]$ is infinite.

This is the remaining case after Cases (1), (2), and (3); and it can be reduced to the discussions in Cases (1), (2), and (3) by considering the maps of fans $\Sigma' \xrightarrow{\varphi} \Sigma_{\varphi} \xrightarrow{\iota_2} \Sigma$ induced by $\varphi : N' \rightarrow N \cap \varphi(N')_{\mathbb{R}} \hookrightarrow N$. This gives a decomposition $\tilde{\varphi} = \tilde{\varphi}_2 \circ \tilde{\iota}_2$. The surjective map $\tilde{\varphi}_2|_{Y'}$ is already discussed in Case (3). Since our goal is to understand the induced fibration of Y' , we may replace $\tilde{\varphi}$ by $\tilde{\varphi}_2$. This concludes our discussion.

In the string literatures, e.g. [K-S1] and [K-S2], Cox homogeneous coordinates are used to understand the fibrations of toric Calabi-Yau hypersurfaces. For completeness of discussion, we recall in the following remark their essential points and fit them into the setting above.

Remark 3.2.4 [Cox homogeneous coordinates]. Assume that $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma}$ is a fibration. Let $\Sigma'(1) = \{v'_1, \dots, v'_{I'}\}$, $\Sigma(1) = \{v_1, \dots, v_I\}$, and $(z'_1, \dots, z'_{I'})$ and (z_1, \dots, z_I) be the corresponding homogeneous coordinates for $X_{\Sigma'}$ and X_{Σ} respectively. Since $\varphi : \Sigma'(1) \rightarrow \Sigma(1)$ is now surjective, $\varphi(v'_i) = r_{ij}z_j$ for some $r_{ij} \in \mathbb{Z}_{\geq 0}$ and φ determines a polynomial map from $\mathbb{C}^{I'}$ to \mathbb{C}^I defined by

$$(z'_1, \dots, z'_{I'}) \longrightarrow (z_1, \dots, z_I) = \left(\prod_i z_i'^{r_{i1}}, \dots, \prod_i z_i'^{r_{iI}} \right).$$

This map is equivariant with respect to the group actions involved and indeed descends to the toric morphism $\tilde{\varphi} : X_{\Sigma'} \rightarrow X_{\Sigma}$. Let Y' be a Σ' -regular (cf. Sec. 4.1) hypersurface in $X_{\Sigma'}$ in the linear system $\mathcal{O}_{X_{\Sigma'}}(D')$, where $D' = \sum_i a_i D_{v'_i}$. Then,

in terms of the given homogeneous coordinates, Y' is described by the zero locus of the polynomial

$$z'^{D'} \sum_{m' \in \Delta' \cap M'} c_{m'} \prod_{i=1}^{|\Sigma'(1)|} z_i'^{m'(v'_i)} = \sum_{m' \in \Delta' \cap M'} c_{m'} \prod_{i=1}^{|\Sigma'(1)|} z_i'^{m'(v'_i) + a_i},$$

where Δ' is the polytope in M' determined by D' . To manifest the fibration $\tilde{\varphi}$, rewrite this polynomial as (cf. Equation (18) in [K-S2])

$$\sum_{\overline{\varphi^\dagger(m')} \in \overline{\Delta'}} c_{\overline{\varphi^\dagger(m')}} \prod_{v'_k \in \varphi^{-1}(0)} z_k'^{\langle \overline{\varphi^\dagger(m')}, v'_k \rangle}$$

with

$$c_{\overline{\varphi^\dagger(m')}} = \sum_{m' \in \overline{\varphi^\dagger(m')}} c_{m'} \prod_{v'_k \notin \varphi^{-1}(0)} z_k'^{m'(v'_k) + a_k}.$$

Formally, this is an equation for the fiber with coefficients $c_{\overline{\varphi^\dagger(m')}}$ regarded as polynomial of coordinates of the base X_Σ .

To realize this explicitly, pick a $\xi : N \rightarrow N'$ as in our discussion of induced morphism $Y' \subset X_{\Sigma'} \rightarrow X_\Sigma$. The map ξ determines a decomposition into three factors :

$$\begin{aligned} & \prod_{v'_k \notin \varphi^{-1}(0)} z_k'^{m'(v'_k) + a_k} \\ &= \prod_{v_j \in \Sigma(1)} z_j^{\langle \xi^\dagger m, v_j \rangle} \cdot \prod_{v'_k \notin \varphi^{-1}(0)} z_k'^{\langle \overline{\varphi^\dagger(m')}, v'_k - \xi \circ \varphi(v_k) \rangle} \cdot \prod_{v'_k \notin \varphi^{-1}(0)} z_k'^{a_k}. \end{aligned}$$

Observe that the first factor is the polynomial in the homogeneous coordinates of X_Σ associated to $\xi^\dagger(m) \in M$ and is the function of the orbit $O_{\{0\}}$ of the base that appear in the proof of Lemma 3.2.1, the last factor is an overall factor that is independent of $\overline{\varphi^\dagger(m')}$, and the middle factor is projected out in this correspondence. This gives a correspondence of our discussion and the ones in homogeneous coordinates for the fiber over $O_{\{0\}}$ as seen in physics literature. This concludes the remark.

Remark. 3.2.5. When $\tilde{\varphi} : Y' \rightarrow X_\Sigma$ is a fibration, though in principle one can write down the equation of the discriminant locus of $\tilde{\varphi}$ but practically it remains very difficult to understand its details (cf. Example 4.2.1).

4 Fibration of Calabi-Yau hypersurfaces via toric morphisms.

The theme of this section is a computational scheme given and demonstrated in Sec. 4.2. Before getting into it, we give a couple of related remarks in Sec. 4.1.

4.1 General remarks.

Remark 4.1.1 [Batyrev]. We recall some definitions and facts about toric hypersurfaces from [Ba2] that will be needed in Sec. 4.2.

- Let Σ'' be a fan in $N_{\mathbb{R}}^{\vee}$. Then the toric orbit decomposition of the ambient toric variety $X_{\Sigma''}$ induces a stratification of a hypersurface Y'' in $X_{\Sigma''}$: $Y'' = \cup_{\sigma'' \in \Sigma''} Y''_{\sigma''}$. Y'' is called Σ'' -regular if each stratum $Y''_{\sigma''}$ is either empty or a smooth subvariety of codimension 1 on $O_{\sigma''}$. In this case, this decomposition is a refinement of the stratification of the hypersurface by the analytic isomorphism classes of singularities. When Σ'' is the normal fan of a polytope Δ' , a Σ'' -regular hypersurface is also called a Δ' -regular hypersurface.
- **Fact [Δ' -regular dense].** ([Proposition 3.1.3 in [Ba2].]) *The set of Δ' -regular hypersurfaces is a Zariski open subset in $\mathbb{P}(L(\Delta'))$, where $L(\Delta')$ is the space of Laurent polynomials with Newton polytope Δ' . For such hypersurfaces, the singularities are induced from the ambient toric variety.*
- The local model for the transverse singularity along the stratum $Y''_{\sigma''}$ of the induced stratification is given by the affine toric variety $U_{\sigma'', N_{\sigma''}^{\vee}}$ associated to the cone σ'' regarded as a fan in $N_{\sigma''}^{\vee} \mathbb{R}$.

Lemma 4.1.2 [regularity]. *Let Σ' corresponds to a maximal projective triangulation of the dual polytope Δ'^* of Δ' , then a Δ' -regular hypersurface in $X_{\Delta'}$ becomes a Σ' -regular hypersurface in $X_{\Sigma'}$ after the associated maximal projective crepant partial (MPCP-)desingularization.*

Proof. Let $\Sigma_{\Delta'}$ be the normal fan of Δ' , which consists of cones over the faces of Δ'^* , and consider the toric morphism $\zeta : X_{\Sigma'} \rightarrow X_{\Delta'}$ associated to the triangulation of Δ'^* . Recall from Proposition 2.1.4 and its proof that the exceptional locus of $\zeta : X_{\Sigma'} \rightarrow X_{\Delta'}$ is a union of the orbit closures of $X_{\Sigma'}$ associated to primitive cones in Σ' with respect to ζ and that, over an orbit $O_{\sigma''}$ of $X_{\Delta'}$, $\zeta^{-1}(O_{\sigma''}) = O_{\sigma''} \times X(\sigma'')$ for some possibly reducible toric variety $X(\sigma'')$ and is the union of toric orbits $\cup_{\sigma' \subset Int(\sigma'')} O_{\sigma'}$ in $X_{\Sigma'}$. Consider now the restriction to hypersurfaces $\zeta : Y' \rightarrow Y''$. For $\sigma' \subset Int(\sigma'')$, $Y'_{\sigma'} = \zeta^{-1}(Y'') \cap O_{\sigma'} = Y''_{\sigma''} \times O_{\sigma', \sigma''}$, where $O_{\sigma', \sigma''}$ is the toric orbit in $X(\sigma'')$ associated to σ' . From this, $Y'_{\sigma'}$ must be empty if $Y''_{\sigma''}$ is empty or smooth of codimension 1 in $O_{\sigma'}$ if $Y''_{\sigma''}$ is smooth of codimension 1 in $O_{\sigma''}$. This concludes the proof. \square

4.2 The computational scheme and a detailed study of an example.

The discussions in the previous subsections together provide us with a toric computational scheme to study the induced fibration of a toric Calabi-Yau hypersurface,

as outlined below :

- The polytope Δ' and the toric variety $X_{\Sigma'}$:
 - The reflexive polytope Δ' .
 - The lattice points in $\Delta' \cap M'$.
 - The dual reflexive polytope Δ'^* in $N'_{\mathbb{R}}$.
 - The triangulation of Δ'^* that gives Σ' .
 - The singular locus of $X_{\Sigma'}$.
- The line bundle $\mathcal{L}_{\Delta'}$ and its restrictions :
 - The restriction of $\mathcal{L}_{\Delta'}$ to $V(\tau')$ for τ' primitive in Σ'_{σ} for some $\sigma \in \Sigma$:
 - * the polytope $\Delta'_{\tau'}$ in $\tau'^{\perp}_{\mathbb{R}}$, up to translation.
 - * the lattice points in $\Delta'_{\tau'} \cap \tau'^{\perp}$.
 - The irreducible component $F_{\sigma}^{\tau'}$ of a generic fiber of $\tilde{\varphi} : V(\tau') \rightarrow V(\sigma)$.
 - The restriction $\mathcal{L}_{F_{\sigma}^{\tau'}}$ of \mathcal{L} to $F_{\sigma}^{\tau'}$:
 - * the polytope $\overline{\Delta'_{\tau'}}$ in $\tau'^{\perp}/\varphi^{\dagger}(\sigma^{\perp})$.
 - * The lattice points of the polytope $\overline{\Delta'_{\tau'}}$ in $\tau'^{\perp}/\varphi^{\dagger}(\sigma^{\perp})$.
- The discriminant locus associated to $\overline{\Delta'_{\tau'}}$ in $\tau'^{\perp}/\varphi^{\dagger}(\sigma^{\perp})$. (I.e. the discriminant locus of the family of sections of the line bundle $\mathcal{L}_{\overline{\Delta'_{\tau'}}}$ over $F_{\sigma}^{\tau'}$.)
- The Σ' -regular sections in $\mathcal{L}_{\Delta'}$ and their restrictions :
 - A Σ' -regular section s of $\mathcal{L}_{\Delta'}$ and its zero-locus, the Calabi-Yau hypersurface Y' .
 - The singular loci of Y' .
 - The restriction of s to $V(\tau')$ for τ' primitive in Σ'_{σ} for some $\sigma \in \Sigma$.
 - The restriction of s to $F_{\sigma}^{\tau'}$:
 - * description of the irreducible components of a generic fiber of $\tilde{\varphi} : Y' \rightarrow X_{\Sigma}$ over each of the toric orbits of X_{Σ} .
 - * restriction of a section of $\mathcal{L}_{\Delta'}$ to $V(\tau')$ and then rewrite it in the fibred form with respect to a map $\xi_{\sigma}^{\tau'} : N/N_{\sigma} \rightarrow N'/N'_{\tau'}$.
- The induced fibration $Y' \rightarrow X_{\Sigma}$ and the stratification of the discriminant locus of the fibration.
- Beyond :
 - Note that before resolving the singularities of Y' , all the fibers, generic and special alike, are described as hypersurfaces of the (perhaps reducible) fiber toric varieties.
 - Fibers after crepant resolution of the singularities of Y' . Note that now some of the fibers may not be realizable as hypersurfaces of fibers of $X_{\Sigma'} \rightarrow X_{\Sigma}$.
 - Monodromy of the fibration.
 - Other details.

The sub-items in Item Beyond are not discussed in general in the current work. We will study some part of it in the specific example below. The purely toric items of this scheme should be programmable, (cf. Problem 5.1 in Sec. 5).

We now apply this to the example of elliptic Calabi-Yau 4-folds in [B-C-dlO-G].

Example 4.2.1 [B-C-dlO-G]. (Continuing Example 2.2.2.) Let us now consider the induced fibration of a 4-dimensional Calabi-Yau hypersurface Y' in $X_{\Sigma'}$ over X_{Σ} from Example 2.2.2. All the notations follow from that example.

- *The polytope Δ' and the toric variety $X_{\Sigma'}$:*

- *The reflexive polytope Δ' :*

- (1) Δ' is the convex hull in $M'_{\mathbb{R}}$ of the following set of 14 generating vertices in M' :

$$\left\{ \begin{array}{ll} m'_1 = (-22, -14, 4, 1, 1), & m'_2 = (-22, 6, 4, 1, 1), \\ m'_3 = (-10, -6, 2, -1, 1), & m'_4 = (-10, 2, 2, -1, 1), \\ m'_5 = (-6, -6, 0, 1, 1), & m'_6 = (0, 0, 0, -2, 1), \\ m'_7 = (0, 0, 0, 1, -1), & m'_8 = (2, -6, 2, -1, 1), \\ m'_9 = (2, 2, 2, -1, 1), & m'_{10} = (6, -14, 4, 1, 1), \\ m'_{11} = (6, -6, 0, 1, 1), & m'_{12} = (6, 3, -3, 1, 1), \\ m'_{13} = (6, 6, -3, 1, 1), & m'_{14} = (6, 6, 4, 1, 1) \end{array} \right\} .$$

- (2) There are 9 codimension-1 faces, i.e. the maximal polytope of the boundary $\partial\Delta'$: (For brevity of notation, only the indices i of the generating m'_i are indicated. Their dual vertices in N' are labelled as subscripts.)

$$\left\{ \begin{array}{ll} [6, 7, 8, 9, 10, 11, 12, 13, 14]_{v'_1}, & [2, 4, 6, 7, 9, 13, 14]_{v'_2}, \\ [3, 4, 6, 7, 8, 9,]_{c'_1}, & [1, 2, 3, 4, 7, 8, 9, 10, 14]_{c'_2}, \\ [1, 2, 5, 7, 10, 11, 12, 13, 14]_{v'_4}, & [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14]_{v'_5}, \\ [1, 3, 5, 6, 7, 8, 10, 11,]_{f'}, & [5, 6, 7, 11, 12]_{g'}, \\ [1, 2, 3, 4, 5, 6, 7, 12, 13]_{v'_6} & \end{array} \right\} .$$

Faces of higher codimensions are obtained by the intersection of subcollections of the above faces.

- *The lattice points in $\Delta' \cap M'$:*

The cardinality $|\Delta' \cap M'| = 3365$. Some sample lattice points in $\Delta' \cap M'$ are

$$\begin{aligned} & (-22, -14, 4, 1, 1), \quad (-22, -13, 4, 1, 1), \quad (-22, -12, 4, 1, 1), \quad \dots \\ & \dots, \quad (0, 0, 0, -1, 1), \quad (0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 1), \quad \dots \\ & \dots, \quad (6, 6, 2, 1, 1), \quad (6, 6, 3, 1, 1), \quad (6, 6, 4, 1, 1). \end{aligned}$$

- *The dual reflexive polytope Δ'^* in $N'_{\mathbb{R}}$:*

- (1) Δ'^* is the convex hull in $N'_{\mathbb{R}}$ of the following set of 9 generating vertices in N' : $\{ v'_1, v'_2, c'_1, c'_2, v'_4, v'_5, f', g', v'_6 \}$.

- (2) There are 14 codimension-1 faces: (Their dual vertices in M' are labelled as subscripts.)

$$\left\{ \begin{array}{ll} [c'_2, v'_4, v'_5, f', v'_6]_{m'_1}, & [v'_2, c'_2, v'_4, v'_5, v'_6]_{m'_2}, \\ [c'_1, c'_2, v'_5, f', v'_6]_{m'_3}, & [v'_2, c'_1, c'_2, v'_5, v'_6]_{m'_4}, \\ [v'_4, v'_5, f', g', v'_6]_{m'_5}, & [v'_1, v'_2, c'_1, v'_5, f', g', v'_6]_{m'_6}, \\ [v'_1, v'_2, c'_1, c'_2, v'_4, f', g', v'_6]_{m'_7}, & [v'_1, c'_1, c'_2, v'_5, f']_{m'_8}, \\ [v'_1, v'_2, c'_1, c'_2, v'_5]_{m'_9}, & [v'_1, c'_2, v'_4, v'_5, f']_{m'_{10}}, \\ [v'_1, v'_2, v'_5, f', g']_{m'_{11}}, & [v'_1, v'_4, v'_5, g', v'_6]_{m'_{12}}, \\ [v'_1, v'_2, v'_4, v'_5, v'_6]_{m'_{13}}, & [v'_1, v'_2, c'_2, v'_4, v'_5]_{m'_{14}} \end{array} \right\}.$$

Faces of higher codimensions are obtained by the intersection of subcollections of the above faces.

- *The triangulation of Δ'^* that gives Σ' :*

- (0) In general, triangulations of a polytope by lattice points can be generated, using the packages "PUNTOS", developed by Loera ([Lo]) and "PORTA", developed by Christof ([Ch1-2] and [Zi]).¹ In this example, the triangulation we use is given in Example 2.2.2 in Sec. 2.2, following [B-C-dlO-V].
- (1) The vertices b' , e'_1 , e'_2 , and e'_3 that lie in the boundary of Δ'^* are added to the vertex set to give a triangulation of Δ'^* , the cones - with apex at the origin - over which form the fan Σ' .
- (2) The relation of b' , e'_1 , e'_2 , and e'_3 with the generating vertices of Δ'^* is given by $b' = (v'_1 + 4c'_1 + v'_6)/6$, $e'_1 = (v'_1 + c'_1 + v'_6)/3$, $e'_2 = (v'_1 + v'_6)/2$, and $e'_3 = (v'_1 + 2v'_5 + v'_6)/4$.

- *The singular locus of $X_{\Sigma'}$:*

- (1) The multiplicity $mult(\sigma')$ of $\sigma' \in \Sigma'$ is in the set $\{1, 2, 3\}$. Since 2 and 3 are primes and Σ' is a simplicial fan, if $mult(\tau') > 1$ and $\tau' \prec \sigma'$, then $mult(\tau') = mult(\sigma')$.
- (2) Cones in Σ' whose multiplicity > 1 but all of whose faces have multiplicity 1 are given by:

$$\begin{aligned} \text{multiplicity 2 : } & v'_5 b'; \\ \text{multiplicity 3 : } & v'_4 b', v'_4 e'_1 e'_2. \end{aligned}$$

From the fan structure of Σ' , the three orbit closures $V(v'_5 b')$, $V(v'_4 b')$, and $V(v'_4 e'_1 e'_2)$ are disjoint from each other. Their union forms the singular locus of $X_{\Sigma'}$. Any other cone with multiplicity > 1 contains exactly one of cones $v'_5 b'$, $v'_4 b'$, $v'_4 e'_1 e'_2$ in its face and hence is contained in the corresponding orbit closure of the latter. (FIGURE 4-2-1.)

¹We would like to thank Shinobu Hosono for providing us with these codes and many helpful guidances of their use.

- (3) The toric geometry of the singular locus and the transverse singularity along it is given below:

- (3.1) $V(v'_5 b')$: A toric 3-variety whose fan is given by Σ (recall the fan of the base). The transverse singularity of $X_{\Sigma'}$ along the open dense orbit of $V(v'_5 b')$ is given by the surface A_1 -singularity.
- (3.2) $V(v'_4 b')$: A toric 3-variety whose fan is also given by Σ . The transverse singularity of $X_{\Sigma'}$ along the open dense orbit of $V(v'_5 b')$ is given by the surface A_2 -singularity.
- (3.3) $V(v'_4 e'_1 e'_2)$: A toric 2-variety whose fan is given by

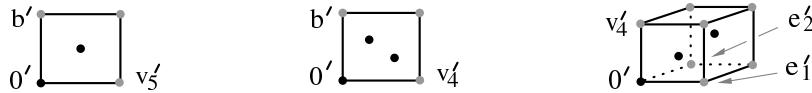
$$\{ (1, 0), (0, 1), (-1, 0), (0, -1) \}.$$

This is $\mathbb{CP}^1 \times \mathbb{CP}^1$. The transverse singularity of $X_{\Sigma'}$ along the open dense orbit of $V(v'_4 e'_1 e'_2)$ is given by the quotient $\mathbb{C}^3 / \mathbb{Z}_3$, where \mathbb{Z}_3 acts on \mathbb{C}^3 by $(z_1, z_2, z_3) \mapsto (\omega z_1, \omega z_2, \omega z_3)$, $\omega^3 = 1$.

- (3.4) Since b' , v'_4 , v'_5 are all mapped to zero under φ , $V(v'_5 b')$ and $V(v'_4 b')$ are indeed two sections of the fibration $\tilde{\varphi}$. They correspond to the two singular points of the generic fiber $W\mathbb{CP}^2(1, 2, 3)$ - one A_1 -singularity and the other A_2 -singularity. Similarly, since the cone $v'_4 e'_1 e'_2$ is mapped to r_1 , $V(v'_4 e'_1 e'_2)$ sits only over $V(r_1)$.

The structure of these orbit closures are obtained from the fact that these cones come from extensions of the 3-dimensional fan Σ'' in Example 2.2.2. The transverse singularities follow from [Ba2] and [Fu] and the symmetry of the fans involved with respect to the lattice structure. Along their lower dimensional orbits, it is the product

$$(\text{the above singularity}) \times \mathbb{C}^{\text{codimension of that orbit}}.$$



- The line bundle $\mathcal{L}_{\Delta'}$ and its restrictions:
 - $\mathcal{L}_{\Delta'}$ has 3365 independent holomorphic sections.
 - The restriction of $\mathcal{L}_{\Delta'}$ to $V(\tau')$ for τ' primitive in Σ'_{σ} for some $\sigma \in \Sigma$:

- * *the polytope $\Delta'_{\tau'}$ in $\tau'^{\perp}_{\mathbb{R}}$, up to translation*: These are the convex hull of a subset of $\{m'_1, \dots, m'_{14}\}$ determined by τ' . They are listed in the $3k + 2$ rows of TABLE 4-2-2.
- * *the lattice points in $\Delta'_{\tau'} \cap \tau'^{\perp}$* : Their numbers are listed in the $3k$ rows of TABLE 4-2-2.

- *The irreducible component F_{σ}' of a generic fiber of $\tilde{\varphi} : V(\tau') \rightarrow V(\sigma)$* : These are given in Example 2.2.2, last column of TABLE 2-2-2. The intersection patterns for these are discussed in that example also, cf. FIGURE 2-2-3.
- *The restriction $\mathcal{L}_{F_{\sigma}'}$ of \mathcal{L} to F_{σ}'* :

* *the polytope $\overline{\Delta'_{\tau'}}$ in $\tau'^{\perp}/\varphi^{\dagger}(\sigma^{\perp})$* : It turns out that, in this example, $\overline{\Delta'_{\tau'}}$ and hence $\mathcal{L}_{F_{\sigma}'}$ depend only on the toric geometry of F_{σ}' . They are given in FIGURE 4-2-3. From the $\overline{\Delta'_{\tau'}}$ worked out, one concludes that except for F_{σ}' that is isomorphic to the generic fiber $\text{WCP}^2(1, 2, 3)$, the restriction $\mathcal{L}_{F_{\sigma}'}$ of $\mathcal{L}_{\Delta'}$ to F_{σ}' is not the anti-canonical line bundle of F_{σ}' . One also observe that the various $\overline{\Delta'_{\tau'}}$ all come from some chopping of the polytope $\overline{\Delta'_{\{0\}}}$ associated to the generic fiber $\text{WCP}^2(1, 2, 3)$.

* *The lattice points of the polytope $\overline{\Delta'_{\tau'}}$ in $\tau'^{\perp}/\varphi^{\dagger}(\sigma^{\perp})$* : The polytopes are indicated in FIGURE 4-2-3. The numbers of lattice points they contain are listed below:

$\text{WCP}^2(1, 2, 3)$	$X(4)$	$\mathbb{C}\text{P}^2$	$X(5)$	$\text{WCP}^2(1, 1, 3)$	\mathbb{F}_2
7	4	6	2	5	4

$O' (54)$	$v'_1 (27)$	$v'_2 (24)$	$c'_1 (16)$
all of m'_i	[6, 7, 8, 9, 10, 11, 12, 13, 14]	[2, 4, 6, 7, 9, 13, 14]	[3, 4, 6, 7, 8, 9]
3365	227	262	154
$c'_2 (12)$	$e'_1 (20)$	$e'_2 (12)$	$e'_3 (16)$
[1, 2, 3, 4, 7, 8, 9, 10, 14]	[6, 7]	[6, 7, 12, 13]	[6, 12, 13]
1242	2	11	10
$f' (16)$	$g' (20)$	$v'_6 (27)$	$f' c'_1 (8)$
[1, 3, 5, 6, 7, 8, 10, 11]	[5, 6, 7, 11, 12]	[1, 2, 3, 4, 5, 6, 7, 12, 13]	[3, 6, 7, 8]
237	64	227	22
$f' c'_2 (6)$	$f' g' (6)$	$g' e'_1 (10)$	$g' e'_2 (6)$
[3, 7, 8, 10]	[5, 6, 7, 11]	[6, 7]	[6, 7, 12]
67	39	2	5
$g' e'_3 (8)$	$v'_2 e'_1 (10)$	$v'_2 e'_2 (6)$	$v'_2 e'_3 (8)$
[6, 12]	[6, 7]	[6, 7, 13]	[6, 13]
4	2	5	4
$v'_2 c'_1 (8)$	$v'_2 v'_2 (6)$	$v'_1 v'_1 (12)$	$v'_1 e'_1 (8)$
[4, 6, 7, 9]	[2, 4, 7, 9, 14]	[6, 7, 9, 13, 14]	[6, 7, 8, 9]
22	86	26	16
$v'_1 c'_2 (6)$	$v'_1 e'_1 (10)$	$v'_1 e'_2 (6)$	$v'_1 e'_3 (8)$
[7, 8, 9, 10, 14]	[6, 7]	[6, 7, 12, 13]	[6, 12, 13]
62	2	11	10
$v'_1 f' (8)$	$v'_1 g' (10)$	$v'_6 v'_2 (12)$	$v'_6 e'_1 (8)$
[6, 7, 8, 10, 11]	[6, 7, 11, 12]	[2, 4, 6, 7, 13]	[3, 4, 6, 7]
19	14	26	16
$v'_6 c'_2 (6)$	$v'_6 e'_1 (10)$	$v'_6 e'_2 (6)$	$v'_6 e'_3 (8)$
[1, 2, 3, 4, 7]	[6, 7]	[6, 7, 12, 13]	[6, 12, 13]
62	2	11	10
$v'_6 f' (8)$	$v'_6 g' (10)$	$v'_1 f' c'_1 (4)$	$v'_1 f' c'_2 (3)$
[1, 3, 5, 6, 7]	[5, 6, 7, 12]	[6, 7, 8]	[7, 8, 10]
19	14	4	6
$v'_1 f' g' (3)$	$v'_1 g' e'_1 (5)$	$v'_1 g' e'_2 (3)$	$v'_1 g' e'_3 (4)$
[6, 7, 11]	[6, 7]	[6, 7, 12]	[6, 12]
7	2	5	4
$v'_1 v'_2 e'_1 (5)$	$v'_1 v'_2 e'_2 (3)$	$v'_1 v'_2 e'_3 (4)$	$v'_1 v'_2 c'_1 (4)$
[6, 7]	[6, 7, 13]	[6, 13]	[6, 7, 9]
2	5	4	4
$v'_1 v'_2 e'_2 (3)$	$v'_6 f' c'_1 (4)$	$v'_6 f' c'_2 (3)$	$v'_6 f' g' (3)$
[7, 9, 14]	[3, 6, 7]	[1, 3, 7]	[5, 6, 7]
6	4	6	7
$v'_6 g' e'_1 (5)$	$v'_6 g' e'_2 (3)$	$v'_6 g' e'_3 (4)$	$v'_6 v'_2 e'_1 (5)$
[6, 7]	[6, 7, 12]	[6, 12]	[6, 7]
2	5	4	2
$v'_6 v'_2 e'_2 (3)$	$v'_6 v'_2 e'_3 (4)$	$v'_6 v'_2 e'_1 (4)$	$v'_6 v'_2 c'_2 (3)$
[6, 7, 13]	[6, 13]	[4, 6, 7]	[2, 4, 7]
5	4	4	6

TABLE 4-2-2. Primitive cones τ' with respect to φ are listed in the $3k + 1$ rows. The toric combinatorial complexity of $V(\tau')$ is indicated by the number of the maximal cones of its fan (in parenthesis). The generating vertices for the polytope $\Delta'_{\tau'}$ are indicated in the $3k + 2$ rows and the number of lattice points contained in it on the $3k$ rows.

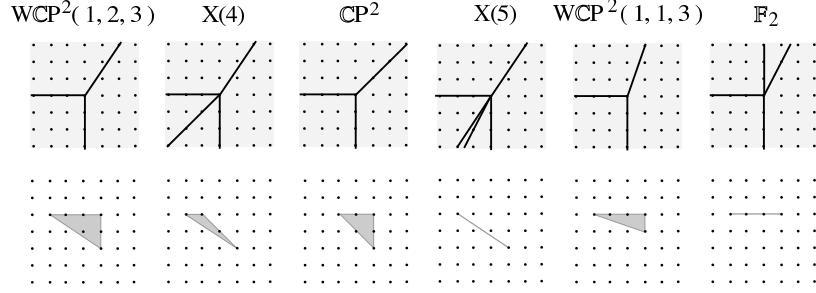


FIGURE 4-2-3. The fan for the six different types of irreducible components $F_\sigma^{\tau'}$ of fibers of $\tilde{\varphi}$ are listed in the first row. The restriction $\mathcal{L}_{F_\sigma^{\tau'}}$ of $\mathcal{L}_{\Delta'}$ to these fibers are described by the polytope $\overline{\Delta'_{\tau'}}$ indicated in the second row. Observe that the fan for $F_\sigma^{\tau'}$ is a refinement of the normal fan of $\overline{\Delta'_{\tau'}}$ and that all the polytopes $\overline{\Delta'_{\tau'}}$ are some chopping of $\overline{\Delta'_{\{0\}}}$.

- The discriminant associated to $\overline{\Delta'_{\tau'}}$ in $\tau'^\perp/\varphi^\dagger(\sigma^\perp)$: They are related to the discriminant locus of the family of sections of the line bundle $\mathcal{L}_{\overline{\Delta'_{\tau'}}}$ over $F_\sigma^{\tau'}$.

- (1) For $\overline{\Delta'_{\tau'}}$ associated to $\text{WCP}^2(1, 2, 3)$: After the integral affine transformation on M' such that $(1, 1) \rightarrow (0, 0)$, $(-2, 1) \rightarrow (3, 0)$, $(1, -1) \rightarrow (0, 2)$, the local polynomial form of a section becomes

$$f(x, y) = a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{01}y + a_{11}xy + a_{02}y^2.$$

Its discriminant is given by ([G-K-Z3] and [La])

$$\begin{aligned} \delta_{\overline{\Delta'_{\tau'}}} = & -27 \cdot 16 a_{00}^2 a_{02}^3 a_{30}^2 - 64 a_{00} a_{02}^3 a_{02}^3 - 64 a_{10}^3 a_{02}^3 a_{30} - 27 a_{01}^4 a_{02} a_{30}^2 \\ & + a_{00} a_{11}^6 + 16 a_{10}^2 a_{20}^2 a_{02}^3 + 16 a_{01}^2 a_{20}^3 a_{02}^2 + a_{02} a_{10}^2 a_{11}^4 - a_{01} a_{10} a_{11}^5 \\ & + a_{01}^2 a_{11}^4 a_{20} - a_{01}^3 a_{11}^3 a_{30} + 32 \cdot 9 a_{00} a_{02}^3 a_{10} a_{20} a_{30} \\ & + 48 a_{00} a_{02}^2 a_{11}^2 a_{20}^2 + 8 \cdot 27 a_{00} a_{01}^2 a_{02}^2 a_{30}^2, -72 a_{01}^2 a_{02}^2 a_{10} a_{20} a_{30} \\ & - 72 a_{00} a_{02}^2 a_{10} a_{11}^2 a_{30} - 16 a_{01} a_{02}^2 a_{10} a_{11} a_{20}^2 - 8 a_{02}^2 a_{10}^2 a_{11}^2 a_{20} \\ & + 96 a_{01} a_{02}^2 a_{10}^2 a_{11}^2 a_{30} - 144 a_{00} a_{01} a_{02}^2 a_{11} a_{20} a_{30} - 12 a_{00} a_{02} a_{11}^4 a_{20} \\ & + 8 a_{01} a_{02} a_{10} a_{11}^2 a_{20} - 8 a_{01}^2 a_{02} a_{11}^2 a_{20}^2 - 30 a_{01}^2 a_{02} a_{10} a_{11}^2 a_{30} \\ & + 36 a_{01}^3 a_{02} a_{11} a_{20} a_{30} + 36 a_{00} a_{01} a_{02} a_{11}^3 a_{30}. \end{aligned}$$

- (2) For $\overline{\Delta'_{\tau'}}$ associated to $X(4)$: After the integral affine transformation on M' such that $(-1, 1) \rightarrow (0, 0)$, $(-2, 1) \rightarrow (0, 1)$, $(-2, 1) \rightarrow (2, 0)$, the local polynomial form of a section becomes $f(x, y) = a_{00} + a_{10}x + a_{20}x^2 + a_{01}y$. Its discriminant is given by ([G-K-Z3]) $\delta_{\overline{\Delta'_{\tau'}}} = 1$.

- (3) For $\overline{\Delta'_{\tau'}}$ associated to $\mathbb{C}\mathbb{P}^2$: After the integral affine transformation on M' such that $(1, 1) \rightarrow (0, 0)$, $(-1, 1) \rightarrow (2, 0)$, $(1, -1) \rightarrow (0, 2)$, the local polynomial form of a section becomes

$$f(x, y) = a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{02}y^2.$$

Its discriminant is ([G-K-Z3])

$$\delta_{\overline{\Delta'_{\tau'}}} = a_{20}a_{01}^2 + a_{10}^2a_{02} + a_{11}^2a_{00} - a_{11}a_{10}a_{01} - 4a_{20}a_{02}a_{00}.$$

- (4) For $\overline{\Delta'_{\tau'}}$ associated to $X(5)$: After an integral affine transformation on M' such that $(-2, 1) \rightarrow (0, 0)$, $(1, -1) \rightarrow (1, 0)$, the local polynomial form of a section becomes $f(x) = a_0 + a_1x$. Its discriminant is given by ([G-K-Z3]) $\delta_{\overline{\Delta'_{\tau'}}} = a_1$.
- (5) For $\overline{\Delta'_{\tau'}}$ associated to $\mathbb{WCP}^2(1, 1, 3)$: After the integral affine transformation on M' such that $(1, 1) \rightarrow (0, 0)$, $(-2, 1) \rightarrow (3, 0)$, $(1, 0) \rightarrow (0, 1)$, the local polynomial form of a section becomes

$$f(x, y) = a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{01}y.$$

Its discriminant is given by ([G-K-Z3]) $\delta_{\overline{\Delta'_{\tau'}}} = 1$.

- (6) For $\overline{\Delta'_{\tau'}}$ associated to \mathbb{F}_2 : After an integral affine transformation on M' such that $(-2, 1) \rightarrow (0, 0)$, $(1, 1) \rightarrow (3, 0)$, the local polynomial form of a section becomes $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Its discriminant is given by ([G-K-Z3]) $\delta_{\overline{\Delta'_{\tau'}}} = 27a_0^2a_3^2 + 4a_0a_2^3 + 4a_1^3a_3 - a_1^2a_2^2 - 18a_0a_1a_2a_3$.

• *The Σ' -regular sections in $\mathcal{L}_{\Delta'}$ and their restrictions:*

- A Σ' -regular section s of $\mathcal{L}_{\Delta'}$ and its zero-locus, the Calabi-Yau hypersurface Y' : The Hodge numbers of these Calabi-Yau hypersurfaces were computed in [B-C-dlO-G]: $h^{1,1} = 8$, $h^{3,1} = 2897$, $h^{2,1} = 1$, and $h^{2,2} = 11662$. The dimension of the polynomial complex moduli space $\mathcal{M}_{\text{poly}}$ is computed by the formula ([Ba2] and [C-K])

$$\dim(\mathcal{M}_{\text{poly}}) = |\Delta' \cap M'| - 6 - \sum_{\Theta'} l(\Theta') = 2897,$$

where Θ' are the codimension-1 faces of Δ' and $l(\Theta')$ is the number of lattice points in the relative interior of Θ' . Thus, the zero-locus of the 3365-parameter family of sections of $\mathcal{L}_{\Delta'}$ gives a 2897-parameter family of Calabi-Yau 4-varieties.

Remark. The triangulation of Δ'^* used so far missed 4 points in $\Delta'^* \cap N'$. The latter come in when $X_{\Sigma'}$ and hence Y' are desingularized. The above counting of moduli is more related to what comes after desingularization.

- *The singular loci of Y' :* Under the assumption that Y' is Σ' -regular, the singular locus $Sing(Y')$ of Y' are induced from that of the ambient $X_{\Sigma'}$. Thus, $Sing(Y') = Y'_{[v'_5 b']} \sqcup Y'_{[v'_4 b']} \sqcup Y'_{[v'_4 e'_1 e'_2]}$. The first two components, if non-empty, have co-dimension 2 in Y' while the last component, if nonempty, has co-dimension 3.
- *The restriction of s to $V(\tau')$ for τ' primitive in Σ'_σ for some $\sigma \in \Sigma$:*
Suppose s is given by

$$s = \sum_{m' \in \Delta' \cap M'} a_{m'} \chi^{m'}, \quad (\text{a total of 3365 summands}).$$

Then

$$s|_{V(\tau')} = \sum_{m' \in \Delta'_{\tau'} \cap M'} a_{m'} \chi^{m'}, \quad (\text{only summands related to lattice points in } \Delta'_{\tau'} \text{ remain}).$$

The character $\chi^{m'}$, $m' \in \Delta'_{\tau'}$, is then rewritten as a character related to the lower dimensional torus associated to $V(\tau')$ by projecting it to τ'^{\perp} . Let us give a concrete example.

Demonstration for $[v'_1, e'_2]$. Let $\tau' = [v'_1, e'_2]$, then

$$\Delta'_{[v'_1, e'_2]} = [m'_6, m'_7, m'_{12}, m'_{13}]$$

(cf. TABLE 4-2-2) and

$$\Delta'_{[v'_1, e'_2]} \cap M' = \left\{ \begin{array}{l} (0, 0, 0, -2, 1), \quad (0, 0, 0, 1, -1), \quad (2, 1, -1, -1, 1), \\ (2, 2, -1, -1, 1), \quad (4, 2, -2, 0, 1), \quad (4, 3, -2, 0, 1), \\ (4, 4, -2, 0, 1), \quad (6, 3, -3, 1, 1), \quad (6, 4, -3, 1, 1), \\ (6, 5, -3, 1, 1), \quad (6, 6, -3, 1, 1) \end{array} \right\}.$$

Fix a basis, e.g. $((2, 0, -1, 1, 0), (0, 1, 0, 0, 0), (6, 0, -3, 0, 2))$, for $[v'_1, e'_2]^{\perp} \cap M'$ and take, say $(0, 0, 0, -2, 1)$, as the origin. Then

$$\Delta'_{[v'_1, e'_2]} \cap M' = \left\{ \begin{array}{l} (0, 0, 0), \quad (3, 0, -1), \quad (1, 1, 0), \quad (1, 2, 0) \\ (2, 2, 0), \quad (2, 3, 0), \quad (2, 4, 0), \\ (3, 3, 0), \quad (3, 4, 0), \quad (3, 5, 0), \quad (3, 6, 0) \end{array} \right\}.$$

and

$$\begin{aligned} s|_{V([v'_1, e'_2])} = & a_{(0,0,0)} + a_{(3,0,-1)} t_1^3 t_3^{-1} + a_{(1,1,0)} t_1 t_2 + a_{(1,2,0)} t_1 t_2^2 \\ & + a_{(2,2,0)} t_1^2 t_2^2 + a_{(2,3,0)} t_1^2 t_3^2 + a_{(2,4,0)} t_1^2 t_2^4 + a_{(3,3,0)} t_1^3 t_2^3 + a_{(3,4,0)} t_1^3 t_2^4 \\ & + a_{(3,5,0)} t_1^3 t_2^5 + a_{(3,6,0)} t_1^3 t_2^6. \end{aligned}$$

To understand the meaning of this, fix a transverse dual basis, say $(0, 0, 0, 1, 0), (0, 1, 0, 0, 0), (0, 0, -1, -1, -1)$, to the cone $[v'_1, e'_2]$ in N' . The 6 maximal cones (cf. Table 4-2-2) in Σ' that contains $[v'_1, e'_2]$ in the face are

$$v'_1 e'_3 e'_1 g' e'_2, \quad v'_1 e'_3 v'_4 g' e'_2, \quad v'_1 e'_1 v'_4 g' e'_2, \quad v'_1 e'_3 e'_1 v'_2 e'_2, \quad v'_1 e'_3 v'_2 v'_4 e'_2, \quad v'_1 e'_1 v'_2 v'_4 e'_2.$$

They project to the following six maximal cones in $Star([v'_1, e'_2])$:

$$\bar{e}'_3 \bar{e}'_1 \bar{g}', \quad \bar{e}'_3 \bar{v}'_4 \bar{g}', \quad \bar{e}'_1 \bar{v}'_4 \bar{g}', \quad \bar{e}'_3 \bar{e}'_1 \bar{v}'_2, \quad \bar{e}'_3 \bar{v}'_2 \bar{v}'_4, \quad \bar{e}'_1 \bar{v}'_2 \bar{v}'_4.$$

where

$$\bar{e}'_1 = (1, 0, 3), \bar{e}'_3 = (0, 0, -1), \bar{v}'_2 = (2, -1, 6), \bar{v}'_4 = (-1, 0, 0), \bar{g}' = (-1, 1, -3).$$

In this way, $\mathcal{L}|_{V([v'_1, e'_2])}$ is described directly by the 3-dimensional polytope : the convex hull of $(0, 0, 0)$, $(3, 0, -1)$, $(3, 3, 0)$, and $(3, 6, 0)$ and $s|_{V([v'_1, e'_2])}$ as given is a combination of characters on the 3-dimensional torus $(\mathbb{C}^\times)^3$ associated to the cone: the origin of the rank 3 lattice in describing $Star([v'_1, e'_2])$.

□

– *The restriction of s to $F_\sigma^{\tau'}$:*

* *description of the irreducible components of a generic fiber of $\tilde{\varphi}$:*
 $Y' \rightarrow X_\Sigma$ over each of the 33 toric orbits of X_Σ :

(1) The essential feature depends only on the toric geometry of the fiber F_σ .

(2) *For $F_\sigma = X(4) \cup \mathbb{CP}^2$:* With labels in FIGURE 2-2-3,

$$\begin{aligned} \mathcal{L}_{c'_1} &= \mathcal{O}(D_1 + D_2 + D_4) \quad \text{and} \quad (D_1 + D_2 + D_4) \cdot D_3 = 2 \quad \text{on } X(4); \\ \mathcal{L}_{c'_2} &= \mathcal{O}(D_5 + D_6) \quad \text{and} \quad (D_5 + D_6) \cdot D_3 = 2 \quad \text{on } \mathbb{CP}^2. \end{aligned}$$

These data restrict the generic degenerate fiber of Y' that lies in $X(4) \cup \mathbb{CP}^2$. (To be continued in Item: Beyond.)

(3) *For $F_\sigma = X(5) \cup W\mathbb{CP}^2(1, 1, 3) \cup \mathbb{F}_2$:* With labels in FIGURE 2-2-3,

$$\mathcal{L}_{e'_1} = \mathcal{O}(D_1 + D_2 - D_3 + D_5) \quad \text{and} \quad \begin{cases} (D_1 + D_2 - D_3 + D_5) \cdot D_3 = 4 \\ (D_1 + D_2 - D_3 + D_5) \cdot D_4 = 0 \end{cases} \quad \text{on } X(5);$$

$$\mathcal{L}_{e'_2} = \mathcal{O}(-D_3 + D_6 + D_7) \quad \text{and} \quad \begin{cases} (-D_3 + D_6 + D_7) \cdot D_3 = 1 \\ (-D_3 + D_6 + D_7) \cdot D_7 = 3 \end{cases} \quad \text{on } W\mathbb{CP}^2(1, 1, 3);$$

$$\mathcal{L}_{e'_3} = \mathcal{O}(-D_7 + D_8 + D_9) \quad \text{and} \quad \begin{cases} (-D_7 + D_8 + D_9) \cdot D_4 = 0 \\ (-D_7 + D_8 + D_9) \cdot D_7 = 3 \end{cases} \quad \text{on } \mathbb{F}_2.$$

These data restrict the generic degenerate fiber of Y' that lies in $X(5) \cup W\mathbb{CP}^2(1, 1, 3) \cup \mathbb{F}_2$. (To be continued in Item: Beyond.)

* *restriction of a section of $\mathcal{L}_{\Delta'}$ to $V(\tau')$ and then rewrite it in the fibred form with respect to a map $\xi_{\sigma'} : N/N_\sigma \rightarrow N'/N'_{\tau'}$:* The details are already explained in the theory part. Let us give an explicit computation here for $[v'_1, e'_2]$ as an example.

Continuing Demonstration for $[v'_1, e'_2]$. Recall that $\varphi : [v'_1, e'_2] \mapsto [d_4, r_1]$. Choose a compatible basis, say, $(d_4, r_1, (0, 1, 0))$ for N and

its dual basis for M . Extend the transverse basis to $[v'_1, e'_2]$ to the full basis, say $(0, 0, 0, 1, 0)$, $(0, 1, 0, 0, 0)$, $(0, 0, -1, -1, -1)$, $(0, 0, 2, 2, 3)$, $(1, 0, 0, -2, -3)$ for N' and take its dual basis for M' . Then (treating vectors as column vectors)

$$\varphi = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and the induced map on quotients $\overline{\varphi}_{d_4 r_1}^{v'_1 e'_2} : N'/N'_{v'_1 e'_2} \rightarrow N/N_{d_4 r_1}$ is given by $\overline{\varphi}_{d_4 r_1}^{v'_1 e'_2} = (0, 1, 0)$ and the dual $\overline{\varphi}_{d_4 r_1}^{v'_1 e'_2 \dagger} : [d_4, r_1]^\perp \rightarrow v'_1 e'_2{}^\perp$ is given by $\overline{\varphi}_{d_4 r_1}^{v'_1 e'_2 \dagger} = (0, 1, 0)^t$. From these, one obtains that

$$\overline{\Delta'_{[v'_1, e'_2]} \cap M'} = \{(0, 0), (1, 0), (2, 0), (3, 0), (3, -1)\},$$

as indicated in FIGURE 4-2-3, up to an integral affine transformation. This describes the line bundle $\mathcal{L}|_{F_{d_4 r_1}^{v'_1 e'_2}}$. The set of cones in $Star([v'_1, e'_2])$ that lie in $\overline{\varphi}_{d_4 r_1}^{v'_1 e'_2 \dagger}(0)$ are given by $\bar{e}'_3 \bar{e}'_1$, $\bar{e}'_3 \bar{v}'_4$, $\bar{e}'_1 \bar{v}'_4$, and their faces. Since $\bar{e}'_1 \rightarrow (1, 3)$, $\bar{e}'_3 \rightarrow (0, -1)$, $\bar{v}'_4 \rightarrow (-1, 0)$ in the 2-dimensional description, this fan describes $WCP^2(1, 1, 3)$. Now choose $\xi_{d_4 r_1}^{v'_1 e'_2} : N/N_{d_4 r_1} \rightarrow N'/N'_{v'_1 e'_2}$ to be, say, $(0, 1, 0)^t$. This induces a rewriting of $s|_{V([v'_1, e'_2])}$ as

$$\begin{aligned} s|_{V([v'_1, e'_2])} &= a_{(0,0,0)} + a_{(3,0,-1)} t_3^3 t_3^{-1} + a_{(1,1,0)} t_1 t_2 + a_{(1,2,0)} t_1 t_2^2 \\ &\quad + a_{(2,2,0)} t_1^2 t_2^2 + a_{(2,3,0)} t_1^2 t_2^3 + a_{(2,4,0)} t_1^2 t_2^4 + a_{(3,3,0)} t_1^3 t_2^3 \\ &\quad + a_{(3,4,0)} t_1^3 t_2^4 + a_{(3,5,0)} t_1^3 t_2^5 + a_{(3,6,0)} t_1^3 t_2^6. \\ &= a_{(0,0,0)} + (a_{(1,1,0)} t_2 + a_{(1,2,0)} t_2^2) t_1 \\ &\quad + (a_{(2,2,0)} t_2^2 + a_{(2,3,0)} t_2^3 + a_{(2,4,0)} t_2^4) t_1^2 \\ &\quad + (a_{(3,3,0)} t_2^3 + a_{(3,4,0)} t_2^4 + a_{(3,5,0)} t_2^5 + a_{(3,6,0)} t_2^6) t_1^3 \\ &\quad + a_{(3,0,-1)} t_1^3 t_3^{-1} \\ &= b_{(0,0)}(t_2) + b_{(1,0)}(t_2) t^1 + b_{(2,0)}(t_2) t_1^2 + b_{(3,0)}(t_2) t_1^3 \\ &\quad + b_{(3,-1)}(t_2) t_1^3 t_3^{-1}. \end{aligned}$$

This is an expression of a section in $\mathcal{L}|_{F_{d_4 r_1}^{v'_1 e'_2}}$ with coefficient functions $b_{(\cdot, \cdot)}(t_2)$ depending on the base parametrized by $t_2 \in \mathbb{C}^\times$.

□

- *The induced fibration $Y' \rightarrow X_\Sigma$ and the stratification of the discriminant locus of the fibration:* Once the section s is X_Σ -orbit-wise written in the form of sections of $\mathcal{L}_{F_\sigma^r}$ with coefficients functions of O_σ , using the discriminant associated to $\overline{\Delta'_{\tau'}}$, one morally can study the stratification of the discriminant

locus of the fibration. Unfortunately, for the current example, the number of parameters involved are so huge that it is above any tool known to us to analyze it.

- *Beyond:* Here we select only *the issue of desingularization* of this example for some discussions.

(1) Recall that

$$\text{Sing}(X_{\Sigma'}) = V([v'_5 b']) \sqcup V([v'_4 b']) \sqcup V([v'_4 e'_1 e'_2]) \simeq X_{\Sigma} \sqcup X_{\Sigma} \sqcup \mathbb{CP}^1 \times \mathbb{CP}^1.$$

To desingularize $X_{\Sigma'}$, one adds

$$\begin{aligned} b'_3 &= (v'_5 + b')/2 = (0, 0, 0, 1, 1), \\ b'_1 &= (v'_4 + 2b')/3 = (0, 0, 0, 1, 2), \quad b'_2 = (2v'_4 + b')/3 = (0, 0, 0, 0, 1), \\ e'_4 &= (v'_4 + e'_1 + e'_2)/3 = (0, 0, 1, 1, 2) \end{aligned}$$

to $\Sigma'(1)$. Since these lattice points lie in Δ'^* , they determine a unique refined triangulation of Δ'^* , which in turn determines a refinement $\widetilde{\Sigma}'$ of Σ' . The following rules tell us how a simplicial cone in Σ' splits to simplicial cones in $\widetilde{\Sigma}'$:

$$\begin{aligned} [\dots, v'_5, b'] &\longrightarrow [\dots, v'_5, b'_3] + [\dots, b'_3, b'] \\ [\dots, v'_4, b'] &\longrightarrow [\dots, v'_4, b'_2] + [\dots, b'_2, b'_1] + [\dots, b'_1, b'] \\ [\dots, v'_4, e'_1, e'_2] &\longrightarrow [\dots, e'_1, e'_2, e'_4] + [\dots, v'_4, e'_2, e'_4] \\ &\quad + [\dots, v'_4, e'_1, e'_4] \end{aligned}$$

All other types of cones remain the same. Denote $\widetilde{X}_{\Sigma'}$ the toric variety associated to the refined fan. One can check that all the new maximal cones have multiplicity 1, thus $\widetilde{X}_{\Sigma'}$ is a smooth 5-fold. Since Y' is Σ' -regular, its proper transform \widetilde{Y}' in $\widetilde{X}_{\Sigma'}$ is also smooth.

- (2) Since $\varphi : b'_3, b'_1, b'_2 \rightarrow 0, e'_4 \rightarrow r_1$, the set of primitive cones over $[\dots, r_1]$ with respect to φ now becomes

$$\{[\dots, e'_1], [\dots, e'_2], [\dots, e'_3], [\dots, e'_4]\}.$$

The primitive cones over all other types of cones in Σ remain the same.

- (3) Following the same argument in earlier parts of the example, one has the following proper transform/resolution of components of fibers

of $\tilde{\varphi}$:

$$\begin{aligned}
 W\mathbb{C}\mathbb{P}^2(1, 2, 3) &\xleftarrow{\quad} \widetilde{W\mathbb{C}\mathbb{P}^2(1, 2, 3)} = X_{\{(1,1), (2,3), (1,2), (0,1), (-1,0), (0,-1)\}} \\
 &\simeq \mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2} \\
 X(4) &\xleftarrow{\quad} \widetilde{X(4)} = X_{\{(1,1), (2,3), (1,2), (0,1), (-1,0), (-1,-1), (0,-1)\}} \\
 &\simeq \mathbb{C}\mathbb{P}^2 \# 4\overline{\mathbb{C}\mathbb{P}^2} \\
 \mathbb{C}\mathbb{P}^2 &\xleftarrow{\quad} \mathbb{C}\mathbb{P}^2 \\
 X(5) &\xleftarrow{\quad} \widetilde{X(5)} = X_{\{(1,1), (2,3), (1,2), (0,1), (-1,0), (-1,-1), (-2,-3), (-1,-2), (0,-1)\}} \\
 &\simeq \mathbb{C}\mathbb{P}^2 \# 6\overline{\mathbb{C}\mathbb{P}^2} \\
 W\mathbb{C}\mathbb{P}^2(1, 1, 3) &\xleftarrow{\quad} \widetilde{W\mathbb{C}\mathbb{P}^2(1, 1, 3)} = X_{\{(1,3), (0,1), (-1,0), (0,-1)\}} \\
 &\simeq \mathbb{F}_3 \\
 \mathbb{F}_2 &\xleftarrow{\quad} \mathbb{F}_2 \\
 \bullet &\xleftarrow{\quad} \mathbb{C}\mathbb{P}^2
 \end{aligned}$$

The intersection relations of these components are induced from the original components and are indicated in FIGURE 4-2-4.

- (4) The restriction of the pullback line bundle on $\widetilde{X_{\Sigma'}}$, still denoted by \mathcal{L} , to these fibers are still described by the same polytopes, except for the new fiber component $\mathbb{C}\mathbb{P}^2$ from the relative stars $Star_{[\dots, r_1]}[\dots', e'_4]$, the restriction of \mathcal{L} to which is trivial. Note that none of the restriction is the anticanonical bundle except for the generic fiber $W\mathbb{C}\mathbb{P}^2(1, 2, 3)$.

(4.1) For $F_{\sigma} = \widetilde{X(4)} \cup \mathbb{C}\mathbb{P}^2$: With labels in FIGURE 4-2-4,

$$\begin{aligned}
 \mathcal{L}_{e'_1} &= \mathcal{O}(D_1 + D_2 + D_4 + E_1 + E_2 + E_3) \\
 \text{and } (D_1 + D_2 + D_4 + E_1 + E_2 + E_3) \cdot D_3 &= 2 \text{ on } \widetilde{X(4)}; \\
 \mathcal{L}_{e'_2} &= \mathcal{O}(D_5 + D_6) \\
 \text{and } (D_5 + D_6) \cdot D_3 &= 2 \text{ on } \mathbb{C}\mathbb{P}^2.
 \end{aligned}$$

(4.2) For $F_{\sigma} = \widetilde{X(5)} \cup \widetilde{W\mathbb{C}\mathbb{P}^2(1, 1, 3)} \cup \mathbb{F}_2 \cup \mathbb{C}\mathbb{P}^2$: With labels in FIGURE 4-2-4,

$$\begin{aligned}
 \mathcal{L}_{e'_1} &= \mathcal{O}(D_1 + D_2 - D_3 + D_5 + E_1 + E_2 + E_3) \\
 \text{and } \begin{cases} (D_1 + D_2 - D_3 + D_5 + E_1 + E_2 + E_3) \cdot D_3 = 1 \\ (D_1 + D_2 - D_3 + D_5 + E_1 + E_2 + E_3) \cdot D_4 = 0 \end{cases} && \text{on } \widetilde{X(5)}; \\
 \mathcal{L}_{e'_2} &= \mathcal{O}(-D_3 + D_6 + D_7) \\
 \text{and } \begin{cases} (-D_3 + D_6 + D_7) \cdot D_3 = 1 \\ (-D_3 + D_6 + D_7) \cdot D_7 = 3 \end{cases} && \text{on } \widetilde{W\mathbb{C}\mathbb{P}^2(1, 1, 3)}; \\
 \mathcal{L}_{e'_3} &= \mathcal{O}(-D_7 + D_8 + D_9) \\
 \text{and } \begin{cases} (-D_7 + D_8 + D_9) \cdot D_4 = 0 \\ (-D_7 + D_8 + D_9) \cdot D_7 = 3 \end{cases} && \text{on } \mathbb{F}_2; \\
 \mathcal{L}_{e'_4} &= \mathcal{O} \text{ on } \mathbb{C}\mathbb{P}^2.
 \end{aligned}$$

- (5) Recall the adjunction formula $g = 1 + \frac{1}{2}(K_S + C) \cdot C$ that computes the genus g of a smooth embedded curve C in the divisor class $[C]$

of a smooth surface S via the intersection number of the canonical divisor K_S of S and the self-intersection number of C . One can check that all the smooth curves in $F_{\sigma}^{\tilde{\tau}'}$ cut out by sections of $\mathcal{L}_{F_{\sigma}^{\tilde{\tau}'}}$, where $\tilde{\tau}'$ is any primitive cone in $\widetilde{\Sigma}'$, are rational, as they should be. Since each type of fibers appear over a codimension-1 stratum of the discriminant locus of the fibration $\widetilde{Y}' \rightarrow X_{\Sigma}$, from the intersection numbers computed in Sub-item (4) above and the Kodaira's table of singular elliptic fibers of a 1-parameter family elliptic fibration, one concludes that: *For the elliptic fibration $\widetilde{Y}' \rightarrow X_{\Sigma}$ of Calabi-Yau 4-fold induced from the toric morphism $X_{\widetilde{\Sigma}'} \rightarrow X_{\Sigma}$, the generic degenerate fiber associated to the $\widetilde{X}(4) \cup \mathbb{CP}^2$ toric fiber has two irreducible components and is of type III (or \widetilde{A}_1) in Kodaira's table while the generic degenerate fiber associated to the $\widetilde{X}(5) \cup W\mathbb{CP}^2(1,1,3) \cup \mathbb{F}_2 \cup \mathbb{CP}^2$ toric fiber has five irreducible components (one in $\widetilde{X}(5)$, one in $W\mathbb{CP}^2(1,1,3)$ and three in \mathbb{F}_2) and is of type I_0^* (or \widetilde{D}_4).*

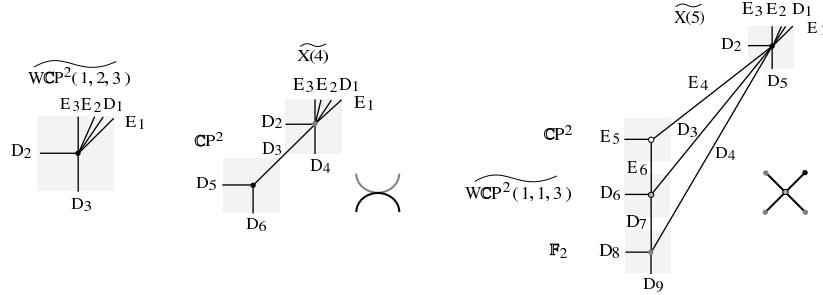


FIGURE 4-2-4. The toric divisors associated to 1-cones in the toric fiber of $\widetilde{X}_{\Sigma'} \rightarrow X_{\Sigma}$ and the Kodaira labels for the generic fiber and the generic degenerate fibers of the elliptic Calabi-Yau hypersurface Y' after toric resolution of the singularities.

This concludes our long discussions of the example. □

5 Remarks on further study.

At the end, we list five problems from our list for further pursuit.

Problem 5.1 [link to Kreuzer-Skarke]. Elaboration of the work of Kreuzer and Skarke ([K-S1] and [K-S2]), which classifies higher dimensional reflexive polytopes that admit fibration, with the computational scheme presented in this paper. This would give further information of fibrations in their list.

Problem 5.2 [fibred Calabi-Yau variety]. The above huge number of concrete examples may provide us a hint for deeper understanding of general fibred Calabi-Yau varieties - toric and non-toric alike -. These examples may also shed some light on the generalization of Kodaira's table of elliptic fibrations to higher dimensional families.

Problem 5.3 [complete intersection]. Extension of the current work to higher rank equivariant vector bundles over a toric variety to take care of fibred Calabi-Yau varieties from complete intersections of hypersurfaces in a toric variety and a toric morphism. Realization of this will complete the first step toward understanding fibred Calabi-Yau variety in the toric category.

Problem 5.4 [effect on quantum cohomology]. In view of the goal that understanding a fibration is meant to understanding the original space in terms of lower dimensional ones and the data how they pile together, one would like to work out a generalized quantum Künneth formula for the quantum cohomology of fibred toric variety (cf. [Ba1] and [K-M]) and also for the hypersurfaces therein. This problem currently looks extremely difficult.

Problem 5.5 [Het/F-theory duality]. Calabi-Yau hierarchy from toric morphisms: a purely toric realization of the Het/F-theory duality. This will be the analogue of the work of Batyrev on mirror symmetry of Calabi-Yau manifolds of the same dimensions. This problem is a final dream of all the participants of the heterotic-string/F-theory duality in the toric category.

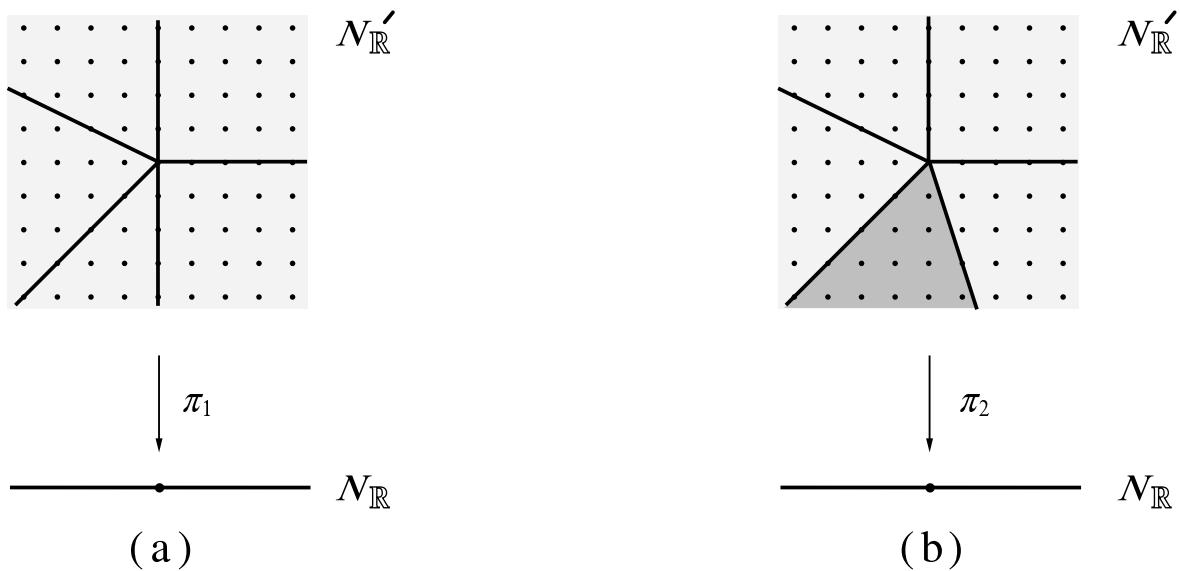
With these problems in mind for further investigations, let us conclude this paper and turn to another new pursuit.

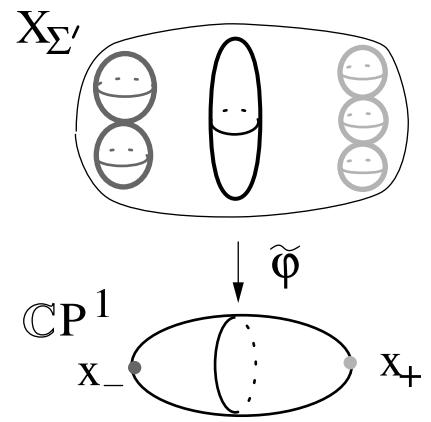
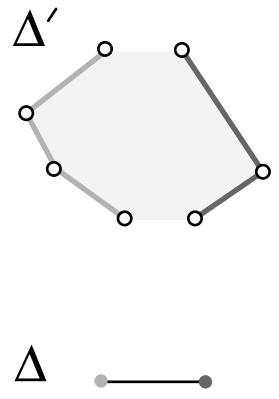
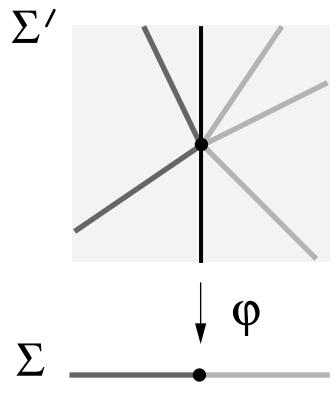
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$W\mathbb{CP}^2(1, 2, 3)$

